Existence, Uniqueness in Nonlinear Systems

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Outline

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Position of the problem

In this part of lectures, we aim to get an understanding about the ordinary differential equation or a differential system expressed as follows :

$$\dot{y} = L(t, y) \tag{1}$$

where L is a continuous map defined :
$(t, y) \mapsto L(t, y)$ from $|a, b| \times \Omega$ to $\mathbb{R}^n$, $|a, b|$ is an interval in $\mathbb{R}$ (open, or closed or semi open or semi closed) Ω is an open set $\mathbb{R}^n$
The problem is to know if there is a map $\mathbb{R}^n$ such that $f = y$ defined from $|a, b|$ from Ω, differentiable and

$$\dot{f}(t) = L(t, f(t)) \tag{2}$$

In this case, we have $\dot{f}(t) \in \mathbb{R}^n$ and even $\dot{f}$ is a continuous function according to the chaine rule theorem of the derivatives.
And then f is continuously differentiable.
A such function is called a solution or an integral of differential equation or differential system.
In the sequel, we are going to call ODE, the differential system. Nevertheless if it is needed, we bring precisions about the type of ODE (scalar or the vector ODE).

Remark

In the real case of $\mathbb{R}^n$, a given function is equivalent to the given of a system composed by n scalar functions $f_1, f_2, \cdots, f_n$.
The map L is equivalent to a system composed by n scalar continuous functions $L_1, L_2, \cdots, L_n$ and the differential system is equivalent to

$$\left\{ \begin{array}{lcl} \dot{y}_1 & = & L_1(t, y_1, y_2, \cdots, y_n) \\ \dot{y}_2 & = & L_2(t, y_1, y_2, \cdots, y_n) \\ \cdots & & \\ \dot{y}_n & = & L_n(t, y_1, y_2, \cdots, y_n) \end{array} \right. \quad (3)$$

An integral or solution of such a differenrial system is then a system of n functions $f_1, f_2, \cdots, f_n$ satisfying

$$\dot{f}_i(t) = L_i(t, f_1(t), f_2(t), \cdots f_n(t)) i = 1, 2, \cdots, n \quad (4)$$

Let us remark that the above ODE are of first order.

Example

Let $X = (x_1, x_2) \in \mathbb{R}^2$,

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = 2x_2 \end{cases}$$

Example

Let $X = (x_1, x_2, x_3) \in \mathbb{R}^3$,

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = x_2 \\ \dot{x}_3 = -x_3 \end{cases}$$

Solving : *It sufifces to integrate each ODE.*

Example

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = 2x_2x_1 \end{cases}$$

To solve this one, one integrates the first ODE and replace x_1 by its expression into the second equation. Solve this latter by separation of the variables.

Example

Mice and Owls or Rabbits and Lenx
Let n be the mouse population depending on time.

$$\dot{n} = r(t)n - k(t)$$

where $r(t)$ is the growth rate, $r(t) \in (0, 1)$ and $k(t)$ is the predation rate, $k(t)$ is a positive number.
Other formulation :

$$\dot{n} = b(t, n)n - d(t, n)n,$$

where b is a function measuring the percentage of birth of the specy and d is a function standing for the death of the specy.

Example

Lokta-Volterra Model
Another classification of differential equations depends on the number of unknown functions that are involved. If there is a single function to be determined, then one equation is sufficient. However, if there are two or more unknown functions, then a system of equations is required. For example, the Lotka-Volterra, or predator- prey, equations are important in ecological modeling. They have the form

$$\begin{cases} \dot{x} &= ax - \alpha xy \\ \dot{y} &= -cy + \gamma xy \end{cases}$$

where $x(t)$ and $y(t)$ are the respective populations of the prey and predator species. The constantsa, α, c, and γ are based on empirical observations and depend on the particular species being studied

Remark

It is important to see that, it is not possible at all to use the method in the previous examples to solve explicitly the last example.s Because there are non linearity terms in the two equations in the Lokta-Volterra equation. And in the other one, even if the equation are simple, we may be stuck. In fact, the classical methods for solving ODE are very limited ! We need to know more : how one can show that a solution exists without computing it explicitly ?

Outline

Overview on some transformations

One can naturally consider ODE of high order. But they can be reduced to ODE of first order.
In fact, let us consider for example the following ODE :

$$y^{(p)} = L(t, y, y^{'}, y^{''}, \cdots, y^{(p-1)}) \tag{5}$$

Here $|a, b|$ is an interval of $\mathbb{R}$,, U is an open set of $\mathbb{R}^n \times \mathbb{R}^{n(p-1)} = \mathbb{R}^{np}$ and L is a continuous application from $|a, b| \times U$ to $\mathbb{R}^n$
One looks for a function f, defined in $|a, b|$ with value functions in $\mathbb{R}^{n(p-1)}$, $p$times differentiable such that :
for every $t \in |a, b|, (f(t), f^{'}(t), f^{''}(t), \cdots, f^{(p-1)}) \in U$ and such that

$$f^{(p)}(t) = L(t, f(t), f^{'}(t), f^{''}(t), \cdots, f^{(p-1)}) \tag{6}$$

Then let us set :

$$f = g_0, f^{'}(t) = g_1, f^{''}(t) = g_2, \cdots, f^{(p-1)} = g_{p-1} \tag{7}$$

We see that finding $\mathbb{R}^n$ is equivalent to the finding of the systems of functions $g_0, g_1, g_2, \cdots, g_{p-1}$ satisfying the following differential system

$$\begin{cases} \dot{g}_0 & = & g_1 \\ \dot{g}_1 & = & g_2 \\ \cdots & & \\ \dot{g}_{(p-2)} & = & g_{(p-1)} \\ \dot{g}_{(p-1)} & = & L(t, g_0, g_1, g_2, \cdots, g_{p-1}) \end{cases} \tag{8}$$

It is easy to remark that this differential system is of first order as the former one but $\mathbb{R}^n$ is replaced by $\mathbb{R}^n \times \mathbb{R}^{n(p-1)}$ and the open set Ω by U.
If one sets $(y_0, y_1, y_2, \cdots, y_{p-1}) = z$, we have to solve

$$\dot{z} = K(t, z), z = g(t) = (g_0(t), g_1(t), \cdots, g_{p-1}(t)) \tag{9}$$

where K is a map defined as follows

$$\begin{array}{ccc} K : |a, b| \times U & \longrightarrow & \mathbb{R}^{np} \\ (t, (y_0, y_1, y_2, \cdots, y_{p-1})) & \mapsto & (y_0, y_1, y_2, \cdots, y_{p-1}, L(t, y, y^{'}, y^{''}, \cdots, y^{(p-1)})) \end{array}$$

Example

$\ddot{x} = sinx + f(x)$, where f stands for a continuous fonction with respect to x.
Solution : Set $\theta = \dot{x}$ and then $\dot{\theta} = \ddot{x} = \sin x + f(x)$. Finally we have

$$\begin{cases} \dot{x} &= \theta \\ \dot{\theta} &= \sin x + f(x) \end{cases} \tag{10}$$

Remark

Let us remark that this transformation is possible if one supposes that the differential equation is regular. This means that the differential equation is solved with respect to the highest order of the derivative. And one has not always this situation in the practice. Many difficulties can occur, and in particular the non existence solutions, or the existence of many solutions corresponding to given initial conditions.
In the next section we are going to present theorems with the assumption that the equation is solved with respect to the highest order of the derivative. And finally, we consider the posed problem under the form (1).

Existence, Uniqueness results

Definition

Cauchy condition
One calls Cauchy condition for a solution of the differential system of form (1), the datum : y_0 of the solution $\mathbb{R}^n$ at a point $t_0 \in |a, b|$.
And sometimes one says " the Cauchy condition t_0, y_0".

Remark

If we have to solve an ODE of ordre p of type (5), a Cauchy condition is expressed in the following sense :
$z_0 = (y_0, y_1, y_2, \cdots, y_{p-1}) = g(t_0)$.

Example

$$\dot{x} = \begin{cases} 1 & \text{if} \quad x < 0 \\ -1 & \text{if} \quad x \geq 0 \end{cases}$$

The vector field on $\mathbb{R}$ points to the left when $x \geq 0$ and to the right of $x < 0$. For the initial condition $x(0) = 0$. Indeed such a solution must initially decrease since $\dot{x}(0) = -1$. But for all negative values of x, solutions must increase. It is impossible. That solutions are never defined for all time : for $x_0 > 0$, the solution is given by $x(t) = x_0 - t$, for $-\infty < t < x_0$. The problem in this example is that the vector field is not continuous at 0.

Definition

One says that L defined in the set $|a, b| \times \Omega$, is locally Lipchitzian in y if : for any $t_0 \in |a, b|$ and for any $y_0 \in \Omega$, there are neighbourhoods U and V of these points and there is $k \in \mathbb{R}_+$ such that for any $t \in U$ and for any $y_1, y_2 \in V$

$$\|L(t, y_1) - L(t, y_2)\| \leq k\|y_1 - y_2\| \tag{11}$$

One says that L defined in $|a, b| \times \Omega$, is globally Lipschitzian in y if there is $k \in \mathbb{R}_+$ such that the condition expressed in (10) is satisfied for any $t \in |a, b|$ and for any $y_1, y_2 \in \Omega$.

Remark

Let us consider $t \geq t_0$ or $t \leq t_0$ real numbers.
Suppose that L is a defined, continuous map from $|a, b| \times \mathbb{R}^n$ in $\mathbb{R}^n$ and globally Lipschitzian in y, it is possible to replace the ODE with the initial condition by an equivalent integral equation.
In fact, let us suppose at first f is a differentiable function and satisfying the initial condition $f(t_0) = y_0$. Then since f and L are continuous functions then $t \mapsto L(t, f(t))$ is continuous too (because of Continuous Chain rule theorem) and then

$$f(t) = y_0 + \int_{t_0}^{t} L(\xi, f(\xi))d\xi. \tag{12}$$

Conversly, if f is a continuous function and solution of the above integral equation, the second right side term is differentiable and its derivative with respect to t is given by :
$t \mapsto L(t, f(t))$ and then f is solution of the ODE, in addition we have $f(t_0) = y_0$.

First Theorems : Successive approximations method : The Picard Iteration Technique.

We suppose that L defined in $|a, b| \times \Omega$, is globally Lipschitzian in y with Lipschitz constant noted k.

Lemma

Let g_1, g_2 be continuous functions defined from $|a, b|$ with value functions in $\mathbb{R}^n$ $n \in \mathbb{N}$ and h_1, h_2 defined by :

$$h_i(t) = y_0 + \int_{t_0}^{t} L(\xi, g_i(xi))d\xi, i = 1, 2.$$

One supposes that : $\exists C \in \mathbb{R}_+$ such that

$$\forall t \in |a, b|, \|g_1(t) - g_2(t)\| \leq C \frac{k^n |t - t_0|^n}{n!}, n \in \mathbb{N}.$$

Then, we have

$$\forall t \in |a, b|, \|h_1(t) - h_2(t)\| \leq C \frac{k^{n+1} |t - t_0|^{n+1}}{(n+1)!} \quad (13)$$

Proposition

Let us consider the ODE (1) with the Cauchy data t_0, y_0. Let us set that

$$\forall t \in |a, b|, y(t) = y_0 + \int_{t_0}^{t} L(\xi, y) d\xi.$$

Let f_0 be a defined function in $|a, b|$ with value function in $\mathbb{R}^n$ (f_0 does not necessarily satisfy the equality $f(t_0) = y_0$, f_0 is arbitrary so that all the writings make sense), for $n \geq 0$, one sets

$$f_{n+1}(t) = y_0 + \int_{t_0}^{t} L(\xi, f_n(\xi)) d\xi. \quad (14)$$

Then, we have

$$\begin{cases} \|f_{n+1}(t) - f_n(t)\| & \leq & C(t)\frac{k^n|t-t_0|^n}{n!} \\ \textit{where} \quad C(t) & \leq & \sup\limits_{\xi\in[t_0,t]} \|f_1(\xi) - f_0(\xi)\| \end{cases} \quad (15)$$

Proposition

Let $f_0, \tilde{f}_0$ be two arbitrary functions, but the wrintings must make sense. Let $f_1, \tilde{f}_1, \cdots, f_n, \tilde{f}_n$ be the functions defined by (13) from the initial data $f_0, \tilde{f}_0$ respectively.
Then

$$\|f_n(t) - \tilde{f}_n(t)\| \leq C(t)\frac{k^n|t - t_0|^n}{n!} \tag{16}$$

where $C(t) = \sup_{\xi \in [t_0, t]} \|f_0(\xi) - \tilde{f}_0(\xi)\|$

We have the following theorem called **Cauchy Lipschitz Theorem : first version**.

Theorem

Existence and uniqueness
Let us suppose a given ODE (1) in which L is a continuous map defined from $|a, b| \times \mathbb{R}^n$ *to* $\mathbb{R}^n$ *and globally Lipschitzian in* y. *Given being a Cauchy condition* t_0, y_0,*the ODE admits a unique solution* f, *defined on* $|a, b|$ *and satisfying the given Cauchy condition* $f(t_0) = y_0$.
In addition we get the following estimation from above of the solution

$$\|f(t)\| \leq e^{k|t-t_0|}\left(\|y_0\| + \int_{t_0}^{t} \|L(\xi, 0)\| d\xi\right) \tag{17}$$

Remark

Some intersting Books :

1. *M. Hirsch, S. Smale, R. L, Devaney, Differential Equations, Dynamical Systems And Introduction to Chaos,*
2. *L. Perko (Differential equations and Dynamical Systems...),*
3. *L. Schwartz (Analyse II : Calcul différentiel et Equations différentielles, ed Herman)*
4. *V. Arnold, Equations différentielles ordinaires, T.1, T.2*

Example

1) $\dot{x} = x; x(0) = x_0 \in \mathbb{R}$. The solution is of course : $x(t) = x_0 exp(at)$. But the aim is to use the Picard iteration technique to get the solution.
We have :

$$y_{k+1}(t) = x_0 + \int_0^t y_k(s)ds, k \in \mathbb{N}^*.$$

And then , we start with $y_0(t) = x_0$ $y_1(t) = x_0 + \int_0^t y_0(s)ds = x_0 + tx_0$. Given y_1, we define

$$y_2(t) = x_0 + \int_0^t y_1(s)ds = x_0 + \int_0^t (x_0 + sx_0)ds = x_0 + tx_0 + \frac{t^2}{2}x_0$$

And so $y_{k+1}(t) = x_0 \sum_{i=0}^{k+1} \frac{t^i}{i!}$.
As $k \to \infty, y_k$ converges to

$$x_0 \sum_{i=0}^{\infty} \frac{t^i}{i!} = x_0 e^t = x(t).$$

Example

2)

$$\begin{cases} \dot{X} = AX \quad \text{where} \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ X(0) = X_0 \end{cases}$$

The solution is given by : $X(t) = exp(tA)X_0$. Let us give some details for $X_0 = (1, 0)$.
For $y_0(t) = (1, 0)$ we have

$$y_{k+1}(t) = X_0 + \int_0^t Ay_k(s)ds, k \in \mathbb{N}^*.$$

And then $y_1(t) = X_0 + \int_0^t Ay_0(s)ds = (1, 0) + \int_0^t (0, -1)ds = (1, -t)$

$$y_2(t) = (1, 0) + \int_0^t Ay_1(s)ds = (1 - \frac{t^2}{2}, -t)$$

Finally ; we have

$$X(t) = \left(\sum_{i=0}^{\infty} (-1)^i \frac{t^{2i}}{(2i)!}, -\sum_{i=0}^{\infty} (-1)^i \frac{t^{2i+1}}{(2i+1)!} \right)$$

On recognizes here that

$$X(t) = (\cos t, -\sin t)$$

Outline

Now suppose that the ODE depends on a parammeter $\lambda \in \Lambda$ be a topological space. One supposes in addition that L is a continuous map on $|a, b| \times \mathbb{R}^n \times \Lambda$ with value function in $\mathbb{R}^n$. We suppose too that L is uniformly Lipschitzian with respect to y.
This means that : $\exists k \geq 0 : \forall (t, \lambda) \in |a, b| \times \Lambda$,(any fixed couple (t, λ)) $\forall y_1, y_2 \in \mathbb{R}^n$, we have the following Lipscitz's inequality

$$\|L(t, y_1, \lambda) - L(t, y_2, \lambda)\| \leq k\|y_1 - y_2\|. \tag{18}$$

Our aim is to explore the following question :
Let's give an initial condition $y_0 = y_0(\lambda), t = t_0(\lambda)$ depending continuously on the parameter λ, does the solution designated by f_λ continuous with respect to λ?
Really, when the initial condition depends on a parameter λ, the solution depends on t and λ.
We can write for each fixed parameter λ, $f_\lambda(t) = f(t, \lambda)$.
And then the ODE and the initial condition are written as follows

$$\left\{ \begin{array}{lllll} (f_\lambda)' & = & \frac{\partial f}{\partial t}(t, \lambda) & = & L(t, f(t), \lambda) \\ f(t_0(\lambda), \lambda) & = & y_0(\lambda) & & \end{array} \right. \tag{19}$$

Theorem

Continuity with respect to a parameter
Let' s consider the ODE (1) (with the right hand side term perturbated as we are going to see soon in thistheorem), a topological space Λ *and a continuous map* $L : |a, b| \times \mathbb{R}^n \times \Lambda \mapsto \mathbb{R}^n$, *uniformly Lipschitzian in* y. *In addition, one supposes that* $\lambda \mapsto t_0(\lambda)$ *and* $\lambda \mapsto y_0(\lambda)$ *are continuous functions from* Λ *to* $|a, b|$ *and from* Λ *to* $\mathbb{R}^n$ *respectively.*
Then if $f : |a, b| \times \Lambda \mapsto \mathbb{R}^n$ *is the unique solution of (18),* $f(t, \lambda)$ *converges to* $f(t, \lambda_0)$ *uniformly on any compact set of* $|a, b|$ *whenever* λ *goes to* λ_0.

Example

$$\begin{cases} \dot{X} = AX \\ X(0) = X_0 = (-1, 0) \end{cases}$$

where $A = \begin{pmatrix} -1 & 0 \\ 0 & k \end{pmatrix}$ The solution is given by : $X(t) = (-e^{-t}, 0)$.
For any $\lambda \neq 0$, let X_λ be the solution satisfying $X_\lambda = (-1, \lambda)$. Then $X_\lambda = (-e^{-t}, \lambda e^{kt})$.
Let us sompute the norm $\|X_\lambda - X(t)\|_1$: it is equal to $|\lambda e^{kt} - 0| = \|X_\lambda(0) - X(0)\|_1 e^{kt} \to 0$ when $\lambda \to 0$,for any fixed time t

Let us extend the approximation method to some Integral equations.
Let $\mathbb{R}^n$ still be a complete normed affine space, Ω be an open set of $\mathbb{R}^n$, $|a,b|$ be an interval of $\mathbb{R}$ and let L be a continuous map defined :
$(x,\xi,y) \mapsto L(x,\xi,y)$ from $|a,b| \times |a,b| \times \Omega$ in $\mathbb{R}^n$.
On the other hand, let $x \mapsto h(x)$ be a continuous function defined from $|a,b|$ in $\mathbb{R}^n$, and finally let x_0 be a point in $|a,b|$. Let us consider the following integral equation

$$f(x) = h(x) + \int_{x_0}^{x} L(x,\xi,f(\xi))d\xi \tag{20}$$

where $\mathbb{R}^n$ is an unknown function and is assumed to be a continuous function from $|a,b|$ in Ω.
This is an integral equation because the unknown $\mathbb{R}^n$ appears under the sign $\int$. It is important to remark that this is a generalization of the formula (11).
The above equation is not equivalent to an ODE. One look for in general a solution $\mathbb{R}^n$ defined on the whole interval $|a,b|$.

Theorem

Let us suppose that there are : a point $y_0 \in \mathbb{R}^n$, a finite number or not $R \geq 0$, a finite number or not $M \geq 0$,a finite number $k \geq 0$ such that the ball $B(y_0, R)$(with center y_0 and radius R) be in Ω and :

1. *for $x \in |a, b|, \xi \in |a, b|, y \in B(y_0, R)$, we have*

$$\|L(x, \xi, y)\| \leq M \tag{21}$$

2. *for $x \in |a, b|, \xi \in |a, b|, y_1, y_2 \in B(y_0, R)$, we have the Lipschitz condition*

$$\|L(x, \xi, y_1) - L(x, \xi, y_2)\| \leq k\|y_1 - y_2\| \tag{22}$$

3. *for any $x \in |a, b|$, we get*

$$\|h(x) - y_0\| + |x - x_0| \leq R. \tag{23}$$

Then
the integral equation (19) gets a unique solution z, taking its values in the ball $B(y_0, R)$ and $z$is obtained by the successive approximation method, uniformly convergent on any bounded interval $|a^{'}, b^{'}|$ in $|a, b|$

Outline

A little more on Existence and Uniqueness Theorems

Definition

Let $J = |t_0 - \alpha, t_0 + \beta|, \alpha > 0, \beta > 0$ be an interval in $|a, b|$, B a closed ball centered in y_0 with finite or not radius R in Ω. J and B are said to a security interval and a security ball respectively (a such couple is said to be a security barrel or a security cylinder) for L, comparatively to $t_0 \in |a, b|$ and $y_0 \in \Omega$, if there is a finite or not number $M \geq 0$ such that,
on the one hand :

$$\|L(t, y)\| \leq M, \forall (t, y) \in J \times B$$

on the other hand, we have the following inequalities

$$\alpha \leq \frac{R}{M}, \beta \leq \frac{R}{M}$$

Remark

A security barrel does always exist. In fact L is assumed to be a continuous function. Then , choosing any couple of points $(t_0, y_0) \in |a, b| \times \Omega$ *there is a neighbourhood* $U_1 = J_1 \times B(y_0, R)$, *such that L is bounded in* U_1*,* ($J_1 =]t_0 - \alpha_1, t_0 + \beta_1[$).
Let us set $M := \sup_{(t,y)\in U_1} \|L(t, y)\|$. *Let us take* $J = |t_0 - \alpha, t_0 + \beta|$ *with* α, β *defined by*

$$\alpha = \min(\alpha_1, \frac{R}{M}), \beta = \min(\beta_1, \frac{R}{M}).$$

And J and B give a good answer to our preoccupation.
In the case where $R = +\infty$ *then* $B = \mathbb{R}^n$ *and we continue to call it a ball.*
And in this case, we can take $M = +\infty$ *and then any finite chosen values for* α *and* β *such that* $|t_0 - \alpha, t_0 + \beta| \subset |a, b|$ *will be suited.*
Even if L is locally Lipschitzian in y, it is still possible to determine the security cylinder, but in a way that L still be Lipschitzian in y.

Theorem

Cauchy- Lipschitz
Given an ODE defined as in (1) where L is a continuous (in time), locally Lipschizian in y defined from $|a, b| \times \Omega$ *in* $\mathbb{R}^n$
Then, for a given Cauchy condition t_0, y_0 *and for a given security cylinder* $J \times B$ *with respect to* t_0, y_0 *such that L be Lipschitzian in y in* $J \times B$*, the ODE (1) admits a unique solution* $\mathbb{R}^n$ *satisfying the Cauchy datum* $f(t_0) = y_0$ *and such that* $f(J) \subset B$*. This solution is got by the successive approximation method :*

$$\left\{ \begin{array}{rcl} f & = & f_0 + (f_1 - f_0) + \cdots + (f_n - f_{n-1}) + \cdots \\ f_n(t) & = & y_0 + \displaystyle\int_{t_0}^{t} L(\xi, f_{n-1}(\xi)) d\xi. \end{array} \right. \tag{24}$$

Example

Let us come back to the Lokta-Volterra model :

$$\left\{\begin{array}{rcl} \dot{x} & = & ax - \alpha xy \\ \dot{y} & = & -cy + \gamma xy \end{array}\right.$$

Since it is not possible to compute explicitly the solution, we have to verify if the hypotheses of the Cauchy Lipschitz theorem are satisfied.Let us recall the norms of a square matrix A of order n : $A = (a_{ij})_{1 \le i,j \le n}$, $a_{ij} \in \mathbb{C}$, then

$$\|A\|_1 = \max_{\|x\|_1=1} \|Ax\|_1 = \max_{j \in \{1,\cdots,n\}} \sum_{i=1}^{n} |a_{ij}|, x \in \mathbb{R}^n$$

$$\|A\|_\infty = \max_{\|x\|_\infty=1} \|Ax\|_\infty = \max_{i \in \{1,\cdots,n\}} \sum_{j=1}^{n} |a_{ij}|, x \in \mathbb{R}^n$$

$$\|A\|_2 = \rho(A^*A)^{\frac{1}{2}}, A^* = \overline{A}^t, \rho(A) = \max\{|\lambda|; \lambda \in Sp(A)\}$$

Let us set

$$L(x,y) = \begin{pmatrix} ax - \alpha xy \\ -cy + \gamma xy \end{pmatrix}$$

The Jacobian matrix of L noted J is :

$$J(x,y) = \begin{pmatrix} a - \alpha y & -\alpha x \\ \gamma y & -c + \gamma x \end{pmatrix}$$

At any point $(x^*, y^*) \in \mathbb{R}^2$; $\|J(x^*, y^*)\|_1$ is bounded !
L is clearly Locally Lipschitzian. Then for a given Cauchy condition $t_0, (x_0, y_0)$ and and for a given security cylinder $J \times B$ with respect to $t_0, (x_0, y_0)$ there is a unique solution of the Lokta-Volterra ODE.

Theorem

Cauchy-Peano
Given an ODE defined as in (1) where L is a continuous, defined from $|a,b| \times \Omega$ *in* $\mathbb{R}^n$*.Then, for a given Cauchy condition* t_0, y_0 *and for a given security cylinder* $J \times B$ *with respect to* t_0, y_0*, the ODE (1) admits at least one local solution f satisfying the Cauchy datum* $f(t_0) = y_0$*.*

Remark

In the case of Cauchy Péano theorem, there is always a solution. That is uniqueness which is the question.

Example

$\dot{x} = 3x^{2/3}; x(0) = 0.$ $x(t) = \begin{cases} 0 & \text{if} \quad t \leq c \\ (t-c)^3 & \text{if} \quad t > c \end{cases}$ where $c > 0$. There is an infinite of solutions ! For each $c > 0$, we have a solution

Theorem

Under the same assumptions than in the Cauchy theorem with $\alpha = \beta$*, for any* $z \in B(y_0, r)$ *and* $s \in I =]t_0 - \eta, t_0 + \eta[$,
$(r = \frac{R}{2}, \eta = \frac{\alpha}{2})$
there is a function $f(s, z, t)$ *defined from* I *in* $B(y_0, R)$*, solution to the following Cauchy problem :*

$$\begin{cases} \dot{y}(t) & = & L(t, y(t)) \\ y(s) & = & z \end{cases} \tag{25}$$

This solution is got by the successive approximation method and it admits the following representation by the series :

$$\begin{cases} f & = & f_0 + (f_1 - f_0) + \cdots + (f_n - f_{n-1}) + \cdots \\ f_n(z, s, t) & = & z + \int_s^t L(\xi, f_{n-1}(z, s, \xi)) d\xi. \end{cases} \tag{26}$$

Outline

Maximum solution and a priori upper bound estimation of solutions

We have just seen that for a chosen time interval J, the successive approximation method succeeds. And it is not necessarily the biggest interval with which the solution of the ODE exists.
In this section our aim is to study the notions of maximal solution and a priori majoration of solutions and the related results to these notions.

Theorem

Under the same assumptions as in the Cauchy theorem, if, in a sub set $|a_1, b_1| \subset |a, b|$, two solutions f_1, f_2 of the ODE (1) take the same value at a point c, these two solutions are the same in all the interval $|a_1, b_1.|$

Corollary

Given the ODE (1) satisfying the conditions of the Cauchy theorem and the Cauchy condition t_0, y_0. Then, there is a maximum interval $|\xi^-, \xi^+|, (a \leq \xi^- \leq t_0 \leq \xi^+ \leq b)$ in which exists one solution of the ODE satisfying the Cauchy condition t_0, y_0.
This solution is unique in this interval, it is extendable until the value ξ^-, (respectively ξ^+) except may be for $\xi^- = a$(respectively may be $\xi^+ = b$).

Definition

One calls maximum solution of the ODE (1), under the hypotheses of the Cauchy theorem ; the unique solution defined on the maximum interval interval $|\xi^-, \xi^+|$ given in the above corollary.

Example

$\dot{x} = 1 + x^2$, the solution is given by $x(t) = tan(t + c), c$ a constant. Such a function cannot be extended over the interval lager than $]-c - \frac{\pi}{2}, c + \frac{\pi}{2}[, x(t) \to \pm\infty$ as $t \to -c \pm \frac{\pi}{2}$

Theorem

Let $U \subset \mathbb{R}^n$ be an open set, and $L : U \to \mathbb{R}^n$ be $\mathcal{C}^1$. Let f be a solution of

$$\dot{f} = L(f)$$

defined on a maximal open interval $J = (\alpha, \beta) \subset \mathbb{R}$ with $\beta < \infty$. Then given any closed and bounded set $K \subset U$, there is some $t \in (\alpha, \beta)$ with $f(t) \notin K$

Remark

This theorem says that if a solution a solution cannot be extended to a larger time interval, then this solution leaves any closed and bounded set in U. This implies that $f(t)$ must arbitrarily close to the boundary of U as $t \to \beta$

Theorem

Let $|a, b|$ be an interval of $\mathbb{R}$, Ω be open set of $\mathbb{R}^n$, L be continuous map and locally Lipschitzian in y, defined from $|a, b| \times \Omega$ in $\mathbb{R}^n$. Let $\mathbb{R}^n$ be the maximum solution of ODE (1) with the Cauchy condition t_0, y_0 defined in the intervals $[t_0, \xi^+[$ with $\xi^+ < b$.
In these conditions, $f(t)$ goes to the boundary of Ω when t goes to ξ^+.
This means : For any compact set of Ω, $f(t) \in K^c$ for $t \geq u$ enough near ξ^+. (u is a fixed number).

Remark

In Theorem 23 and Corollary 24 the extension of the solution is an important question for ODE theory. We submit this question as follows : Let $\alpha < t < \beta$ *,* $L(t, x(t))$*, be the vector field,* $x(t) \in O$ *an open set of* $\mathbb{R}^n$ *satisfying the minimal regularity as in Theorem 18 and* $y(t)$ *be the solution of the ODE associated to this vector field. Let us note by D the domain where L is defined (*$D \subset (\alpha, \beta) \times O$*).*
Then $y(t)$ *is not extendable iif one of the following items is satisfied*

1. $\beta = +\infty$ *(resp.* $\alpha = -\infty$*);*
2. $\lim_{t \to \beta^-} \|y(t)\| = +\infty$ *(resp.* $\lim_{t \to \alpha^+} \|y(t)\| = +\infty$*);*
3. *the distance from the point* $(t, y(t))$ *to the border of D goes to zero when* $t \to \beta^-$ *(resp* $t \to \alpha^+$*).*

Theorem

Let us consider the following scalar ODE

$$\dot{z} = M(t, z) \tag{27}$$

M is a continuous, non negative real valued function defined on $[t_0, \xi^+] \times \mathbb{R}_+$.
Let fbe a differentiable function defined from $[t_0, \xi^+]$ *in* $\mathbb{R}^n$ *verifying*

$$\|\dot{f}\| < M(t, \|f(t) - O\|), or \|\dot{y}\| < M(t, \|y - O\|), \tag{28}$$

where O isa chosen origin point in $\mathbb{R}^n$.
If $f(t_0) = y_0$ *and if g defined in* $[t_0, \xi^+]$*is a non negative solution of (26), corresponding to the initial conditions* $t_0, \|y_0 - O\|$, *then we have : for any* $t \in [t_0, \xi^+]$ *the following upper bound estimation*

$$\|f(t) - O\| \leq g(t) \tag{29}$$

The above inequality is strict whenever $t > t_0$. *If one replaces the interval* $[t_0, \xi^+]$ *by* $[\xi^-, t_0]$, *one must to replace g by a non negative solution h of the equation* $\dot{z} = -M(t, z)$ *corresponding to the initial conditions,* $t_0, \|y_0 - O\|$.

Corollary

a version of Gronwall Lemma
Let $[t_0, t_0 + \alpha[$ *an interval of* $\mathbb{R}(\alpha > 0)$, *u a continuous non negative value function defined in that interval. One supposes that for any* $t \in t_0, t_0 + \alpha[$ *, we have the following inequality*

$$u(t) \leq \gamma + \int_{t_0}^{t} (\mu u(s) + \nu) ds, (\mu > 0, \gamma, \nu \in \mathbb{R}), \tag{30}$$

then for any $t \in t_0, t_0 + \alpha[$, *we have the following upper bound estimation*

$$u(t) \leq (\gamma + \frac{\nu}{\mu}) e^{\mu(t-t_0)} - \frac{\nu}{\mu} \tag{31}$$

Outline

Existence of global solutions

Theorem

Let us consider the ODE (1) and suppose that L gets the two following properties :

1. *L is defined from* $]a,b[\times\mathbb{R}^n$ *and one has the below estimation from above*

$$\|L(t,y)\| \leq \mu\|y-O\|+\nu \tag{32}$$

   where μ *and* ν *are non negative constants, O is a chosen origin in* $\mathbb{R}^n$

2. *For any positive number* $\rho(\rho>0)$, *there is a number* $k=k(\rho)$, *such that, whenever t varies in* $]a,b[$ *and* y_1, y_2 *vary in the ball* $B(O,\rho)$, *we have the globally Lipschitz condition.*

Then, for any initial condition t_0, y_0, *the ODE (1) admits a unique solution, defined on the whole interval* $]a,b[$

Remark

About the derivative of highest order of the solution of ODE : if the map L is of class $\mathcal{C}^m, m \in \mathbb{N}$ then the solution of the ODE (1) is of classe $\mathcal{C}^{m+1}$, if L if $\mathcal{C}^{+\infty}$ then the solution of the ODE (1) is $\mathcal{C}^{+\infty}$.

Outline

Notions of Vector field and flow in $\mathbb{R}^n$:

Vector field

Let's consider

$$\dot{x}(t) = L(t, x(t)), x(t_0) = x_0 \tag{33}$$

for any state z the vector $L(t, z)$ gives the variation of the solution $x(t)$ when $x(t) = z$. Then $z \mapsto L(t, z)$ is a vector- valued function. It is called **vector field** (I would like to point out that it is quite possible to define rigorously the notion of vector field as it is done in differential geometry but in this lecture we barrow the more or less approach of slight explanations for this concept). It is a family of vectors depending on t and such that $L(t, z)$ has its origin in z.

Remark

Note that the regularity of L implies that one can talk about the variation of L with respect to $t \in I \subset \mathbb{R}$, (I is an interval) or to $z \in U \subset \mathbb{R}^n$ (I is an open set). And conversely a vector field defines an ODE. And then one says that the solution of the ODE are integral curves of the vector field.

Remark

1. *It is quite possible to set the same definition if U is an open set of a differentiable manifold $\mathcal{M}$. In this case, a vector field f satisfies $f(t,z) \in T_z U$, that is the tangence condition to $\mathcal{M}$.*
2. *If $L(t,z)$ is a vector field, an integral curve $y(t)$is, at every time t, tangent to this vector field. Then on can see that the solving of an ODE is closely linked to a problem formulated in geometric terms.*

Definition

Let $L(t, z)$ be a vector field defined on $I \times U$ (I is a interval of $\mathbb{R}$ and $U \subset \mathbb{R}^n$ is open). One calls an equilibrium (or a singular) point associated to L, any point $z \in U$ such that $L(t, z) = 0$ for any $t \in I$.

Remark

The equilibrium points are the invariant states (z) by the evolution of the system ($\dot{z} = 0$)

Definition

An ODE is said to be autonomuous if it is definned by a stationary vector field i.e $\dot{x} = L(x)$.

Remark

Every non autonomuous ODE can be transformed into an autonomuous one. In fact let us set $y = (s, x)$ *then* $y(t) = (s(t), x(t))$. *And the ODE becomes*

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & L(s(t), x(t)) \\ \dot{s} & = & 1 \end{array} \right. \tag{34}$$

We deduce from the second equation of the above system that $s(t) = t + c$ *where* c *is a constant. If we suppose that* $s(t_0) = t_0$ *where* t_0 *is any time* $t_0 \in I$, *then* $s(t) = t$.
Setting $F(s, x) = (L(s, x), 1)$, *we have :*

$$\dot{x} = L(t, x) \Longrightarrow \dot{y} = F(y).$$

And we increase one dimension in the state space and an additinal condition $s(t_0) = t_0$.

Remark

The state space stands for all the possible states ($x(t)$) that are solutions of the ODE. But it is not sufficient, from an initial position, to predict the future of the system (the future positions).
*Now if we expand the space by taking into account the positions and the velocities, then, from an initial position it is quite possible to predict the future of the system. This space is called **the phase space.***

Definition

Let $y^{(m)} = L(t, y, \cdots, y^{(m-1)})$ an explicit ODE of order m where y is a function with vector valued in $\mathbb{R}^n$;
then the phase space associated to ODE is $X = \{(y, y_1, \cdots, y_{m-1})\} \subset \mathbb{R}^{nm}$.

Flow

In this section we begin by the notion of **change of variables** which consists in changing the system of coordinates. In this case, the ODE is transformed and one interesting thing is to see how it is transformed and its consequences. We suppose that the change of variables is done by a diffeomorphism ϕ, then $X = \phi(x), x = \psi(X), \psi = \phi^{-1}$.
The ODE $\dot{x} = L(t, x)$ is transformed as follows :

$$\dot{X}(t) = \frac{d}{dt}\phi(x) = D\phi(x(t))\frac{dx}{dt}$$

$$= D(\phi(x(t)))L(t, x) = D\phi(\psi(X(t)))L(t, \psi(X(t))).$$

For any fixed t the above last expression in the right hand side is the image of the vector L by the linear map $D\phi$. The evaluation is done at point $(t, \psi(X(t)))$. Hence we get a new ODE and a new vector field. This latter is called the vector field image of L by ϕ.

Example

Consider $\ddot{x} = 0$ in $\mathbb{R}$.
One introduces $E = \{(x, v)\} = \mathbb{R}^2$ the phase space where one rewrites the above equation :

$$\dot{x} = v, \dot{v} = 0.$$

Let us consider the following change of variables :

$$\phi(x, v) = (x + v, x^3) =: (\phi_1, \phi_2).$$

While one does not reach $x = 0$, ϕ defines a diffeomorphism. Let us write x and v with respect to ϕ : $x = \phi_2^{1/3}$, $\quad v = \phi_1 - v = \phi_1 - \phi_2^{1/3}$. To find the transformation of the equation, one calculates :
$\frac{d}{dt}(x + v) = \dot{x} + \ddot{x} = \dot{x} = \phi_1 - \phi_2^{1/3}$, $\frac{d}{dt}x^3 = 3x^2\dot{x} = 3\phi_2^{2/3}(\phi_1 - \phi_2^{1/3})$. In these new variables the system is written as follows :

$$\dot{\phi} = \left(\phi_1 - \phi_2^{1/3}, 3\phi_2^{2/3}(\phi_1 - \phi_2^{1/3})\right).$$

This is equivalent to the initial equation $\ddot{x} = 0$, admittedly ... but the reader will say that it is not evident !

Definition

Let L be a vector field defined on a open set of $\mathbb{R}^n$ and ϕ be a diffeomorphism from U to an open set V, of $\mathbb{R}^m$ then one defines $\phi_* L$, the vector field image of L by ϕ as follows :

$$\phi_* L(X) = D\phi(\phi^{-1}(X))L(\phi^{-1}(X))$$

Let 's now consider an ODE with an initial condition (a Cauchy problem) in the phase space. Then one can correspond to the Cauchy problem a trajectory describing the evolution of the system. This correspondance, which to the initial condition, is associated to the trajectory, is called the **flow** of the ODE. It depends on the initial time t_0.

Definition

Let an ODE defined on a interval I and let X the associated phase space. Let $t_0 \in I$ be the initial time. One calls associated **flow**, the map as follows :

$$\Phi : x_0 \mapsto (x(t))_{t\in I}, \text{ solution to the ODE such that}$$

$$x(t_0) = x_0.$$

Remark

VERY IMPORTANT

- *The flow depends on the initial time t_0, and we will note it $\Phi_{t_0,t}(x_0) = x(t)$ (this is understood as follows : " the state at time t of the system that at time t_0 is in x_0" or $\Phi_{t_0} : x_0 \mapsto (x(t))_{t\in I}$).*
- *The flow is defined only when the ODE gets solution on the whole interval I. Unfortunately, it is not possible in general to hope a such strong result. Then, we are obliged to work with flows partially defined. And the notion of maximum flow is introduced that is the flow defined on a time interval as great as possible.*
- *There is a difference between the trajectory and the flow. The trajectory is the a function $t \mapsto x(t)$ that describes the successive positions of the system. While the flow is the map that to an initial condition is associated the whole trajectory.*

Remark

VERY IMPORTANT *Furthermore, the flow is defined on the phase space, while the trajectory takes its values in the state space smaller than the phase state : for instance, considering a scalar ODE of second order, it is quite possible to study the trajectory in the state of positions. But there does not make sense to talk about the flow in this space of positions.*
In fact to talk about the flow for a scalar ODE of second order, the hint is to transform this equation into a system of first order and the flow in $\mathbb{R}^2$ *or its subset.*

- *The flow satisfies the semi group property :*

$$\Phi_{t_1,t_2} o \Phi_{t_0,t_1} = \Phi_{t_0,t_2}$$

- *The flow is invertible :* $\Phi_{t,t_0} \, o \, \Phi_{t_0,t} = Id$
- *In the case of autonomous system, the flow depends only on* $t - t_0 : \Phi_{t_0,t} = \Phi_{t-t_0}$ *and* $\Phi_{t+s}(x_0) = \Phi_t(\Phi_s(x_0))$.
  There is U a neighbourhood of x_0 *such that* $\Phi_{-t}(\Phi_t(x)) = x \ \forall x \in U$ *and* $\Phi_t(\Phi_{-t}(y)) = y, \forall y \in \Phi_t(U) = V$, *V is an open set.*

Example

$\dot{x} = ax, x(0) = x_0 \in \mathbb{R}$, then $x(t) = x_0 exp(at)$. And $\Phi_t(x_0) = x_0 exp(at)$

Example

$\dot{x} = Ax$ where A is a square matrix of order n, $\Phi_t(x_0) = exp(tA)x_0, \Phi_0 = Id$.

Comments :
With the flow, one can set questions such as : If there is a mistake on the initial condition, is that this will lead to a significant error on the solution ? Will be the error tend to decrease with the time, or to amplify ? Is that the answer to such a question depends crucially on the choice of the original data, or not too ?

Definition

A dynamical system on $\mathbb{R}^n$ is a function

$$\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

where $\phi(t, X) = \phi_t(X)$ satisfies

1. $\phi_0 : \mathbb{R}^n \to \mathbb{R}^n$ is the identity function $\phi_0(X) = X$;
2. The composition $\phi_t \, o \, \phi_s = \phi_{t+s}$ for each $t, s \in \mathbb{R}$.

See the above two examples.

But

Remark

$\ddot{x} + sinx = \epsilon sin2\pi t$ is an example where, for $M = \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$ we have $\phi_t \, o \, \phi_s \neq \phi_{t+s}$! In fact, this ODE is equivalent to

$$\begin{cases} \dot{x} & = & \theta \\ \dot{\theta} & = & -\sin x + \epsilon \sin 2\pi t \end{cases} \tag{35}$$

Let us set $L_\epsilon(t, x, \theta) = (\theta, -\sin x + \epsilon \sin 2\pi t)$. This system is non autonomous. And with the periodicity equal to 1 fin t we have

$$\Phi_{t_0}^{t_0+n} = \Phi_{t_0+n-1}^{t_0+n} o \Phi_{t_0+n-2}^{t_0+n-1} o \cdots o \Phi_{t_0}^{t_0+1} = (\Phi_{t_0}^{t_0+1})^n.$$

The dynamical system $(M, f_\epsilon = \Phi_{t_0}^{t_0+1})$ in descrete time in $\mathbb{Z}$ is :

$$(n, x) \mapsto f_\epsilon(x)^n.$$

It describes orbits in time $t_0 + n, n \in \mathbb{Z}$ given by $\Phi_{t_0}^{t_0+n} = f_\epsilon(x)^n$.

Let us put again in another equivalent version in $\mathbb{R}^n$.

Theorem

Extension Theorem

Let O be an open bounded set of $\mathbb{R}^n$, I be a bounded time interval, $t_0 \in I$ and $L = L(t, x)$ a vector field defined in O, locally Lipschitzian and continuous in time. Then, there is a unique maximum flow $(\Phi_{t_0,t}(x_0))_{t\in I(x_0)}$ for the ODE $\dot{x} = L(t, x(t))$. Furthermore, the flow is locally Lipshcitzian with respect to x_0 and its diffential is given by

$$\begin{cases} \dot{X}(t) &= DL(t, x(t))X(t) \\ X(t_0) &= Id \end{cases} \tag{36}$$

In addition, let $x_0 \in O$, t^* one end (bound) of the interval $I(x_0)$ so that (t^*, x_0) should be at the border of the doamin D; then there a accumulation point y^* of $(\Phi_{t_0,t}(x_0))_{t\in I(x_0)}$ when t goes to t^*, such that $(t^*, y^*) \in \partial(I \times O)$;
where :
D is a subset of $I \times O$ called the domain of the flow (where it makes sense) and verifying :

1. for any $x_0 \in O$, $I(x_0) = \{t \in I; (t, x_0) \in D\}$;is an open interval on which $x(t) = \Phi_{t_0,t}(x_0)$ is solution of $\dot{x} = L(t, x(t))$;
2. There is no map $\Phi_{t_0,t}(x_0)$ satisfying the same properties and defined on a domain greater than D.

Remark

In Theorem 18 (is called also Cauchy- Lipschitz Theorem), the flow $\Phi_{t_0,t}(x_0)$ is defined in a unique way for any time close to t_0 such that the flow makes sense. And it is locally Lipschitz with respect to x_0 and continuous in time. And if the vector field L is of regularity $\mathcal{C}^r$, $r \in \mathbb{N}^*$ with respect to x and continuous in time then $\Phi_{t_0,t}(x_0)$ is $\mathcal{C}^r$, $r \in \mathbb{N}^*$ regular with respect to x_0. But the Hölder regularity is not sufficient to define a flow even for an irregular flow !

Example

- $\dot{x} = a(t)x, x(0) = x_0$ where a is a continuous function. The solution is given by :

$$x(t) = x_0 \int_0^t a(s)ds$$

Example

-

$$\begin{cases} \dot{x}_1 = x_1 + x_2^2 \\ \dot{x}_2 = -x_2 \end{cases}$$

Theorem

Theorem of recovery

Let $g \in \mathcal{C}^1(O, \mathbb{R}^n)$ be a vector field O is an open set of $\mathbb{R}^n$, $y_* \in O$ such that $g(y_*) \neq 0$. Then, there is a diffeomorphism ϕ, defined on a neighbourhood of y_*, such taht $\phi_* g = \xi$ where ξ is constant vector field.
Remind that $\phi_* g$ is the vector field image of g.

Remark

This theorem is equivalent to the Cauchy- Lipschitz theorem.

Conservation laws and Lyapunov function

Definition

Let an ODE, and X be the associated phase space. One call an **invariant** or **conservation law** of the ODE, any function $h : X \to \mathbb{R}$ that remains constant along the solutions of ODE : that means

$$\frac{d}{dt}h(x(t)) = 0.$$

Otherwise, $t \mapsto h(x(t))$ is constant on the interval I where x is defined.

Definition

Let an ODE and X be the associated phase space. Let $h : X \to \mathbb{R}$ a function, h is said to be a **Lyapunov** function if it is either decreasing along the solutions of ODE or increasing along the solutions of this one.

Definition

One calls **gradient flow** of E, a differentiable function defined on $\mathbb{R}^n$, the flow associated to the following ODE :

$$\dot{y} = -\nabla E(y).$$

In this case, we have $\frac{dE(y)}{dt} = \dot{y}.\nabla E(y) = -(\nabla E(y))^2 \leq 0$. *Interpretation :* In fluid mechanics, Finance, Biology, Economy, in Physics ....

Example

An example for the conservation law or an invariant, we consider the Lokta-Volterra model. One considers $I(x,y)$ and then we look it for such that $\frac{dI}{dt} = \dot{x}\frac{\partial I}{\partial x} + \dot{y}\frac{\partial I}{\partial y} = 0$. On choice is $I(x,y) = F(x) + G(y)$.And then replacing in the above equality and using the separation of variables, , we have

$$(ax - \alpha xy)\frac{dF}{dx} + (-cy + \gamma xy)\frac{dG}{dy} = 0.$$

This is equivalent to

$$(x\frac{dF}{dx})\frac{1}{-c + \gamma x} = -(y\frac{dG}{dy})\frac{1}{a - \alpha y} = const.$$

After integration we have

$$I(x,y) = \alpha y + \gamma x - a\ln y - c\ln x,$$

Notion of divergence

From the derivative of $x(t)$ with respect to the initial condition, we can deduce the important tool that is the **divergence**. Let us consider

$$\dot{Z}(t) = DL(t, x(t))Z(t), Z(t_0) = Id$$

For $t > t_0$, the matrix $Z(t)$ indicates how the flow has done vary the space of states from t_0 to t.In particular the eigenvalues give us valuable information on the way which the distances were distorted by the flow (contracted for the eigenvalues such that the modulus are less than 1) or stretched (for the eigenvalues such that the modulus is greater than 1).
One can be interested by the transformation of the volume by the flow. In this case the jacobian determinant : $det(\frac{dx}{dx_0}) = detZ(t)$ will give us the information.
In summary, $det(D\phi)$ is the coefficient by which the transformation ϕ multiplies the infinitesimal volumes. Let's set $\Gamma(t) = detZ(t)$,one can show that

$$\frac{d\Gamma}{dt} = \Gamma(t)tr(DL(t, x(t))).$$

The quantity $tr(DL(t, x(t)))$ is called the **divergence** of the vector field L, and it is noted $\nabla.L = divL$.

Theorem (Liouville's Theorem)

Suppose
$\nabla . L = 0$. *Then for any region* D_0, $V(t) = V(0)$, *where* $V(0)$ *is the volume of* D_0 *and* $V(t)$*is the volume of* $\Phi_t(D_0)$.

Example

a case of *Liouville's Theorem*
A Hamiltonian system is a conservation system in $\mathbb{R}^2$.

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial y} \\ \dot{y} = -\frac{\partial H}{\partial x} \end{cases}$$

Then we have $\frac{dH}{dt} = 0$. Take $L = (\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x})$

Outline

Euler Method or tangent line method :

Recall two important facts about the first order initial value problem

$$\dot{x} = L(t, x(t)), x(t_0) = x_0. \tag{37}$$

First, if L and $\dfrac{\partial L}{\partial x}$ (The Jacobian Matrix) are continuous, then the initial value problem (3) has a unique solution $x = \phi(t)$ in some interval surrounding the initial point $t = t_0$.
Second, it is usually not possible to find the solution ϕ by symbolic manipulations of the differential equation. Up to now we have considered the main exceptions to this statement, namely, differential equations that are linear, separable, or exact, or that can be transformed into one of these types. Nevertheless, it remains true that solutions of the vast majority of first order initial value problems cannot be found by analytical means such as those considered in the first part of this chapter.
Therefore it is important to be able to approach the problem in other ways. As we have already seen, one of these ways is to draw a direction field for the differential equation (which does not involve solving the equation) and then to visualize the behavior of solutions from the direction field. This has the advantage of being a relatively simple process, even for complicated

However, it does not lend itself to quantitative computations or comparisons, and this is often a critical shortcoming.
Another alternative is to compute approximate values of the solution $x = \phi(t)$ of the initial value problem (3) at a selected set of t-values.
Ideally, the approximate solution values will be accompanied by error bounds that assure the level of accuracy of the approximations.
Today there are numerous methods that produce numerical approximations to solutions of differential equations. Here, we introduce the oldest and simplest such method, originated by Euler about 1768. It is called the tangent line method or the Euler method.
Let us consider how we might approximate the solution $x = \phi(t)$ of equation (3) near $t = t_0$. We know that the solution passes through the initial point (t_0, x_0) and, from the differential equation, we also know that its slope at this point is $L(t_0, x_0)$. Thus we can write down an equation for the line tangent to the solution curve at (t_0, x_0), namely,

$$x = x_0 + L(t_0, x_0)(t - t_0). \tag{38}$$

The tangent line is a good approximation to the actual solution curve on an interval short enough so that the slope of the solution does not change appreciably from its value at the initial point.

Thus, if t_1 is close enough to t_0, we can approximate $\phi(t_1)$ by the value x_1 determined by substituting $t = t_1$ into the tangent line approximation at $t = t_0$; thus

$$x_1 = x_0 + L(t_0, x_0)(t_1 - t_0). \tag{39}$$

To proceed further, we can try to repeat the process. Unfortunately, we do not know the value $\phi(t_1)$ of the solution at t_1. The best we can do is to use the approximate value x_1 instead. Thus we construct the line through (t_1, x_1) with the slope $L(t_1, x_1)$,

$$x = x_1 + L(t_1, x_1)(t - t_1). \tag{40}$$

To approximate the value of $\phi(t)$) at a nearby point t_2, we use (6) instead, obtaining

$$x_2 = x_1 + L(t_1, x_1)(t_2 - t_1).$$

Continuing in this manner, we use the value of x calculated at each step to determine the slope of the approximation for the next step.
The general expression for x_{n+1} in terms of t_n, t_{n+1} and x_n is

$$x_{n+1} = x_n + L(t_n, x_n)(t_{n+1} - t_n), n = 0, 1, 2, ... \tag{41}$$

If we introduce the notation $L_n = L(t_n, x_n)$, then we can rewrite equation (6) as

$$x_{n+1} = x_n + L_n(t_{n+1} - t_n), n = 0, 1, 2, ... \tag{42}$$

Finally, if we assume that there is a uniform step size h between the points $t_0, t_1, t_2, \cdots$, then $t_{n+1} = t_n + h$ for each n and we obtain Euler' s formula in the form

$$x_{n+1} = x_n + L_n h, n = 0, 1, 2, \cdots \tag{43}$$

To use Euler' s method you simply evaluate equation (8) or equation (9) repeatedly, depending on whether or not the step size is constant, using the result of each step to execute the next step. In this way you generate a sequence of values $x_1, x_2, x_3, \cdots$ that approximate the values of the solution $\phi(t)$ at the points $t_1, t_2, t_3, ...$

If, instead of a sequence of points, you need an actual function to approximate the solution $\phi(t)$, then you can use the piecewise linear function constructed from the collection of tangent line segments.

That is, let x be given by equation (4) in$[t_0, t_1]$, by equation (6) in [t1, t2], and, in general, by

$$x = x_n + L(t_n, x_n)(t - t_n), n = 0, 1, 2, ... \tag{44}$$

in $[t_n, t_{n+1}]$.

The Euler Method :
Several Steps :

1. define $L(t, x)$
2. input initial values t_0 and x_0
3. input step size h and the number of the steps n
4. output t_0 and x_0
5. for j from 1 to n do $k_1 = L(t, x)$
   $y = y + h * k_1$
   $t = t + h$
6. output t and y
7. end.

Remark : There is another variation of the Euler Method and the estimation of errors in the numerical approximations : cf for instance Chapter 8 of the book : W. E. Boyce and R. C. Diprima : Elementary Differential Equations and Boundary Value Problems, J. Wiley and Sons, Inc, 2001.

An Improved Euler Formula named also the Heun formula :

$$x_{n+1} = x_n + \frac{L(t_n, x_n) + L(t_{n+1}, x_{n+1})}{2} h$$

with

$$x_{n+1} = x_n + hL(t_n, x_n.)$$

For the details cf Chapter 8 of the book : W. E. Boyce and R. C. Diprima : Elementary Differential Equations and Boundary Value Problems, J. Wiley and Sons, Inc, 2001.

Runge Kutta method

The Euler and improved Euler methods belong to what is now called the Runge Kutta1 class of methods. The Runge Kutta formula involves a weighted average of values of $L(t,x)$at different points in the interval $t_n \leq t \leq t_{n+1}$. It is given by

$$x_{n+1} = x_n + (\frac{k_{n1} + 2k_{n2} + 2k_{n3} + k_{n4}}{6})h,$$

where,

$$k_{n1} = L(t_n, x_n) \qquad k_{n2} = L(t_n + \frac{h}{2}, x_n + \frac{hk_{n1}}{2}), \tag{45}$$

$$k_{n3} = L(t_n + \frac{h}{2}, x_n + \frac{hk_{n2}}{2}) \qquad k_{n4} = L(t_n + \frac{h}{2}, x_n + \frac{hk_{n3}}{2}). \tag{46}$$

(47)

The sum $\frac{k_{n1}+2k_{n2}+2k_{n3}+k_{n4}}{6}$ can be interpreted as an average slope.
For the details cf for instance Chapter 8 of the book : W. E. Boyce and R. C. Diprima : Elementary Differential Equations and Boundary Value Problems, J. Wiley and Sons, Inc, 2001.

Multisteps Methods

As recall one consider the initial value problem

$$\dot{x} = L(t, x), x(t_0) = x_0$$

in which data at the point $t = t_n$ are used to calculate an approximate value of the solution $x(t_{n+1})$ at the next mesh point $t = t_{n+1}$. In other words, the calculated value of $x(t) = \phi(t)$ at any mesh point depends only on the data at the preceding mesh point. Such methods are called one-step methods. However, once approximate values of the solution $x = \phi(t)$ have been obtained at a few points beyond t_0, it is natural to ask whether we can make use of some of this information, rather than just the value at the last point, to calculate the value of $\phi(t)$ at the next point. Specifically, if x_1 at t_1, x_2 at t_2, . . . , x_n at tn are known, how can we use this information to determine x_{n+1}at t_{n+1}? Methods that use information at more than the last mesh point are referred to as multistep methods. **There are two types of multistep methods : Adams methods and backward differentiation formulas.** Within each type one can achieve various levels of accuracy, depending on the number of preceding data points that are used. For details cf for instance Chapter 8 of the book : W. E. Boyce and R. C. Diprima : Elementary Differential Equations and Boundary Value Problems, J. Wiley and Sons, Inc, 2001

IMPORTANT Remarks : Stress on how to use these methods by Matlab Software, or Scilab software or any other software

For the Matlab Software :

- Getting Starts and use for instance the books :
  - Brian R. Hunt Ronald L. Lipsman Jonathan M. Rosenberg with Kevin R. Coombes, John E. Osborn, and Garrett J. Stuck : A Guide to MATLAB for Beginners and Experienced Users, Cambridge University Press, 2001
  or
  - Jean Louis Merrien Analyse Numérique avec Matlab : Rappels, Méthodes, Exercices et problèmes avec corrigés détaillés ed Dunod, Paris 2007.

Thank you for your attention