Dynamical Systems: Adding lecture

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Outline

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In this part , we intend to introduce some additional notions on discrete dynamical systems. It is followed by some important parts such as the conjugacy, the stability of the structure and the bifurcation theory.

1- Discrete Dynamical Systems :

1-1 Definitions Proprieties and Consequences :

Definition

A discrete time invertible dynamical system is a group of transformations defined by a bijection map $\{\Phi^n : \Gamma \longrightarrow \Gamma\}_{n\in\mathbb{Z}}$, where Γ represents the phase space, n is the parameter.
A phase space is a structure corresponding to a set of all possibles states ("needed") of the considered system.
A discrete time non invertible dynamical system is a semi group of transformations defined by a map $\{\Phi^n : \Gamma \longrightarrow \Gamma\}_{n\in\mathbb{N}}$,$n$ is the parameter.

Orbits

One calls direct orbit or forward orbit in x_0 and one notee $\mathcal{O}^+(x_0)$ the set : $\{\varphi^n(x_0)\}_{n\in\mathbb{N}}$.
One calls fixed point of a dynamical system any point x_0 such that $\varphi(x_0) = x_1 = x_0$. In this case $\mathcal{O}^+(x_0) = \{x_0\}$.

A concept that generalizes the notion of fixed point is the periodic point notion. Let us assume that there is $k \in \mathbb{N}, k \geq 1$ such that $x_k = x_0$ then one calls that a point x_0 is periodic of period k.
If $k \geq 2$ and $x_k = x_0$ but $x_i \neq x_0, \quad \forall i \in \{2, \cdots k-1\}$ then x_0 is a periodic point with a minimal period equal to k.
$x_{2k} = \varphi^k(x_k) = \varphi^k(x_0) = x_0.$
If x_0 is a periodic with minimal period equals $k \geq 1$ then
$\mathcal{O}^+(x_0) = \{x_0, x_1, \cdots, x_{k-1}\}.$
x_0 is a pre-periodic point if there is $j \in \mathbb{N}, j \geq 1$ such that x_j is a periodic point of period k . In this case $\mathcal{O}^+(x_0) = \{x_0, x_1, \cdots, x_j, \cdots, x_{k+j-1}\}, x_j = x_{k+j}.$

Remark

If φ is invertible, one can define $\mathcal{O}^-(x_0) = \{x_{-n}\}_{n\in\mathbb{N}} = \{\varphi^{-n}(x_0)\}_{n\in\mathbb{N}}$.

Remark

Limit sets
When x_0 is neither periodic nor pre- periodic then $\mathcal{O}^+(x_0)$ is an infinite set. And it's description is more complicated. In general, it is impossible to describe this set without additional hypotheses on the structure, say when the phase space is a topololigical space. And even in this case, the description is not an easy task in many situations.

Example

Let us consider $x_n = (x_n^1, x_n^2)$. Let A lbe the matrix defined as follows :

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$$

$x_{n+1} = Ax_n$ is a discrete time linear system. The problem is to find x_n?

Example

Let us consider the sequence of vectors defined by $v_n = (x_n, y_n, z_n)$ where $v_0 = (x_0, y_0, z_0)$. Let the following system

$$\begin{cases} x_n &= x_{n-1} + z_{n-1} \\ y_n &= x_{n-1} + y_{n-1} \\ z_n &= y_{n-1} + z_{n-1} \end{cases}$$

This is equivalent to

$$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \\ z_{n-1} \end{pmatrix}$$

2 Conjugacy :

Definition

Let X and Y, be two sets , $f : X \to X$ and $g : Y \to Y$, two maps. One says that f and g are conjugate if :
$\exists h : X \to Y$ a bijection such that $hof = gof$.

Example

1. Let A be a diagonalizable matrix, then , there exists a passage matrix P such that the diagonal matrix D is given by the following formulae :

$$D = P^{-1}AP.$$

   In this case on says that D and A are conjugate.

2. Let $A = \begin{pmatrix} -1 & -3 \\ -3 & -1 \end{pmatrix}$ et $B = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}$, there is a rotation matrix such that i $B = R^{-1}AR$. $R = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ should be.
   And then , A and B are conjugate.

Lemma

1. *The conjugacy relation is an equivalence class.*
2. *Let f, g, h as in the above definition, then we have :*
   $f^n = (h^{-1} ogoh)^n = h^{-1} og^n oh$

Remark

The conjugacy relation is to be understood in the sense of equivalence between two dynamical systems.

Lemma

If f and g are two conjugate maps, then they are the same number of periodic points and these are mapped to each other by the conjugacy.

Definition

Let $S : Y \to Y$ and $T : X \to X$.
One says that the dynamical system (Y, S) is a factor of the dynamical system (X, T) if tehre is an onto map $\phi : X \to Y$ verifying

$$\phi\, o\, T = S\, o\, \phi.$$

It is said also that (X, T) is an extension of (Y, S). One says also that ϕ is a semi- conjugacy between S and T.

Outline

3- Stable framework :

An interesting question is the effect of perturbation of dynamical systems with respect to this equivalence relation. In fact when there is equivalence (conjugacy relation), how, a small perturbation will behave between to equivalent systems ? One way to formulate this, is to ask whether two dynamical systems which are "close" in some sense, belong to the same equivalence class.
To formalise this idea we need to formalise the notion of a small perturbation, i.e. we need a topology on a given space of dynamical systems.

Let us attempt to say more : Let ϕ_t, ψ_t two equivalent dynamical systems defined on a phase space X, $\phi_t : X \to X$ and $\psi_t = h^{-1} o \phi_t o h$ with $h : X \to X$. Let's denote by ϕ_t^ϵ the perturbed dynamical system, with ϵ a very small parameter.
The question is : what is the behavior of ϕ_t^ϵ when ϵ goes to zero ?
In general there may be many such topologies and an appropriate topology will depend on the class of systems of interest and on the kind of properties that are being studied. However, assuming some space $\chi$of dynamical systems has been fixed, together with some equivalence relation $\sim$ and some topology τ. we can make the following definition.

Definition

We say that a dynamical system $f \in \chi$ is structurally stable with respect to the topology τ and the equivalence relation $\sim$ if it lies in the interior of its equivalence class. If it is not structurally stable we say that it is (or undergoes) a bifurcation.

Definition

Stability of a point :

1. The equilibrium x_0 is Lyapunov stable (or simply stable) if, for each $\epsilon > 0$and $t_0 \geq 0$, there exists$\delta = \delta(t_0, \epsilon) > 0$ such that $|x(t_0)| < \delta \Longrightarrow x(t)$ exists for all $t \geq t_0$ and $|x(t)| < \epsilon$.
2. The equilibrium $x_0$is uniformly stable if $\delta = \delta(\epsilon)$.
3. The equilibrium x_0 is unstable if it is not stable.
4. The equilibrium x_0 is asymptotically stable if
   (a) x_0 is stable ;
   and
   (b) $\exists \delta_0 = \delta_0(t_0)$such that $|x(t_0)| < \delta_0 \Longrightarrow |x(t)| \to 0$ at $t \to \infty$.
5. The equilibrium x_0 is uniformly asymptotically stable (u.a.s) if
   (a) x_0 is uniformly stable ;
   and
   (b) $\exists \delta_0 > 0$ such that,for each $\epsilon > 0$, there exists $T = T(\epsilon) > 0$ such that $|x(t_0)| < \delta_0, t \geq t_0 + T \Longrightarrow |(x(t))| < \epsilon$.

3- 1 Global study :

- Extension theorem

Corollary

A flow locally bounded on $\mathbb{R}^n$ is globally defined.
Let $L = L(t,x)$ be a vector field of class C^1, defined on $\mathbb{R} \times \mathbb{R}^n$. Let Φ be the maximum flow associated to the ODE $\dot{x} = L(t,x)$, from $t_0 = 0$. If it is known that for any (x_0, t) such that Φ be defined we have,

$$|t| \leq T, \ , |x_0| \leq R \Longrightarrow |\Phi_{t_0,t}(x_0)| \leq C(T,R) < +\infty,$$

then the flow is defined for all the times and all the initial conditions.

- Hartman-Grobman Theorem, Stable manifold theorem, Center Manifold Theorem.

- Flows near a periodic orbit : Stable Manifold Theorem for Periodic Orbits, Poincaré's Stability Condition in the case of two dimensions, Bendixson' s criterion in the case of two dimensions, Dulac' s Criteria in the case of two dimensions, Poincaré- Bendixson Theorem in the case of two dimensions. *For additional information about these notions see for instance the books of L. Perko, J. Hale., Hirsch-Smale book,...*

Definition

A ponit $p \in U \subset \mathbb{R}^n$ (an open set) is a $\omega-$limit point of the trajectory $\phi(., x_0)$ of the system

$$\dot{x} = L(x)(**)$$

if there is a sequence $t_n \to +\infty$ such that $\lim_{n\to} \phi(t_n, x_0) = p$.
Similary if there is a sequence $t_n \to -\infty$ such that $\lim_{n\to+\infty} \phi(t_n, x_0) = q$ and the point q is called an $\alpha-$limit point of the trajectory $\phi(., x_0)$ of (**). The set of all $\omega-$limit points of a trajectory Γ is called the $\omega-$limit set of Γ and it is denoted by $\omega(\Gamma)$.
The set of all $\alpha-$limit point of a trajectory Γ is called $\alpha-$limit set of Γ and it is denoted by $\alpha(\Gamma)$.
$\alpha(\Gamma) \cup \omega(\Gamma)$ is called the limit set of Γ.

Definition (ω-limit sets)

One calls the set ω-limit of a trajectory $(x(t))$ the set of all its accumulation points when $t \to \infty$, i.e

$$\Omega := \Big\{ y;\ \exists (t_n) \to \infty;\ x(t_n) \to y \Big\}.$$

In other words, it is the set of all the possible limits of successive positions of the trajectory evaluated for t tending to ∞.

Theorem

$\alpha(\Gamma)$ and $\omega(\Gamma)$ are closed subsets of $U \subset \mathbb{R}^n$ (an open set) and if Γ is contained in a compact subset of $\mathbb{R}^n$, then $\alpha(\Gamma)$ and $\omega(\Gamma)$ are non empty connected, compact subsets of U.

Definition

A closed invariant set $A \subset U$, an open set of $\mathbb{R}^n$ is called an attracting set of (**) if there is some neighbourhood V of A such that for all $x \in V$, $\phi_t(x) \in V$ for all $t \geq t_0$ and $\phi_t(x) \to A$ as $t \to \infty$.
An attractor of (**) is an attracting set which contains a dense orbit.

Definition (Trajectories linking equilibria)

Let $(x(t))_{t\in\mathbb{R}}$ be a trajectory of an ODE linking two equilibria points :

$$\lim_{t\to-\infty} x(t) = x_-^*, \qquad \lim_{t\to+\infty} x(t) = x_+^*.$$

One says that x is a non homocline trajectory if $x_+^* \neq x_-^*$, and a homocline one if $x_+^* = x_-^*$.

A first version of Poincaré-Bendixson theorem is :

Theorem

Assume that $n = 2$. Let K be a nonempty compact limit set (of $\dot{x} = f(x)$). If K contains no equilibria, then K is a periodic orbit.

Another more general statement of the Poincaré-Bendixson theorem :

Theorem (Poincaré–Bendixson Theorem)

Let us consider an autonomuous ODE of first order with a vector field of class C^1, in an open domain O of $\mathbb{R}^2$. Then, in every compact subset of O containing a finite number of equilibrium points, the $\omega-$ limit sets of an orbit of the ODE can only be one of the three following objects :
- an equilibrium ;
- a cycle ;
- a graph composed by a finite number of unstable equilibrium points connected by trajectories (homoclines or nonhomoclines) of the flow. In addition, two different equilibrium are connected by at most two nonhomoclines, one in each direction of the time. (This graph may include an arbitrary and even a denumerable infinite of homoclines trajectories from a given equilibrium).

Outline

4-Bifurcation :

This part has been prepared with helpful of the Perko 's book and lecture notes given by Michael Y. Li in December 7, 2004 .

4- 1 Introduction :

Consider a family of differential equations

$$\dot{x} = f(x, \mu), \quad (1)$$

where $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is analytic for $\mu \in \mathbb{R}, x \in \mathbb{R}^n$.
Let $x_0(\mu)$be a family of equilibria of (1), namely, $f(x_0(\mu), \mu) = 0$.
Set $z = x - x_0(\mu)$.
Then,
$\dot{z} = A(\mu)z + O(|z|^2), A(\mu) = Df(x_0(\mu), \mu)$.
Let $\lambda_1(\mu), \cdots, \lambda_n(\mu)$ be the eigenvalues of $A(\mu)$. **If, for some** $i, i \in \{1, \cdots, n\}; \mathcal{R}e\lambda_i(\mu)$ **changes sign at** $\mu = \mu_0$, **we say that** μ_0 **is a bifurcation point of (1). Sometimes, we also call** $(x_0(\mu_0), \mu_0)$ **at bifurcation point.**

Remark

1. *f is analytic in x, μ implies that $x_0(\mu)$ is analytic in μ , provided $detA(\mu) = detDf(x_0(\mu), \mu) \neq 0$. Analyticity may fail at a bifurcation point since $detA(\mu) = \lambda_1(\mu) \cdots \lambda_n(\mu)$.*
2. *Being the roots of $det(\lambda I - A(\mu)) = 0, \lambda = \lambda_i(\mu)$ are also analytic in μ except possibly at bifurcation points.*

4- 2 In one dimension :

Since only one-dimensional eigenspace changes with μ, we may simply assume $n = 1$.
Therefore, $f : \mathbb{R}^2 \to \mathbb{R}$, and $x_0(\mu)$ is a real-valued analytic function of μ provided

$$\lambda_1(\mu) = \frac{\partial f}{\partial x}(x_0(\mu), \mu) = A(\mu) \neq 0.$$

Therefore, the equilibrium $x_0(\mu)$ is u.a.s. if $\lambda_1(\mu) < 0$, and unstable if $\lambda_1(\mu) > 0$.
This implies that μ_0 is a bifurcation point if

$$\lambda_1(\mu_0) = 0.$$

Therefore, bifurcation points $(x_0(\mu_0), \mu_0)$ are solutions of

$$f(x, \mu) = 0, \text{ and } \frac{\partial f}{\partial x}(x, \mu) = 0 \tag{2}$$

The bifurcation diagram describes the general shape of $x_0(\mu)$ for μ near the bifurcation point μ_0.
At $\mu \neq \mu_0, \lambda_1(\mu) = \frac{\partial f}{\partial x}(x, \mu) \neq 0$. By the Implicit Function Theorem,$x = x_0(\mu)$ is the unique solution of

$$f(x, \mu) = 0,$$

and

$$\frac{\partial f}{\partial x}\frac{dx}{d\mu} + \frac{\partial f}{\partial \mu} = 0.$$

Since
$\lambda_1(\mu_0) = \frac{\partial f}{\partial x}(x_0(\mu_0), \mu_0) = 0$, if $\frac{\partial f}{\partial \mu}(x_0(\mu_0), \mu_0) \neq 0$, then $|\frac{dx}{d\mu}| \to \infty$, as $\mu \to \mu_0$. Therefore the curve $x = x_0(\mu)$ has a vertical tangent line when $\mu = \mu_0$ if $\frac{\partial f}{\partial \mu}(x_0(\mu_0), \mu_0) \neq 0$. In the subsequent discussions, without loss of generality (w.l.o.g.), we will assume that the bifurcation point is at $(0, 0)$, namely, $\mu_0 = 0$ and $x_0(\mu_0) = 0$.
The most common bifurcation types are illustrated by the following examples.

Example

Saddle-Node Bifurcation. Consider

$$\dot{x} = \mu - x^2.$$

In this case, the bifurcation equations (2) becomes

$$\begin{aligned} x^2 &= \mu, \\ 2x &= 0. \end{aligned}$$

There are two branches of equilibria :
$x_0(\mu) = \pm\sqrt{\mu}$, and the bifurcation point is at $(x, \mu) = (0, 0)$.
Since $\frac{\partial f}{\partial \mu}(0, 0) \neq 0$, we should expect a vertical tangent line at $(0, 0)$ for $x_0(\mu)$.

Example

Transcritical Bifurcation. Consider

$$\dot{x} = \mu x - x^2.$$

The bifurcation equation (2) becomes $\mu x - x^2 = 0, \mu - 2x = 0$.
This gives two branches of equilibria $x_0 = 0$, and $x_0 = \mu$.
For the branch $x_0 = 0$, we have $\lambda_1 = \mu$ and and thus the stability changes from stable to unstable as μ increases cross 0, and $\mu_0 = 0$ is the bifurcation point.
For the second branch, $\lambda_1 = -\mu$. Therefore, this branch changes stability in the opposite direction to the first branch, and the bifurcation point is also $\mu_0 = 0$.

Example

Pitchfork Bifurcation. Consider

$$\dot{x} = \mu x - x^3.$$

From the bifurcation equations, we find that there are three branches of equilibria : $x_0 = 0, x_0 = \sqrt{\mu}$ and $x_0 = -\sqrt{\mu}$.
The corresponding λ_1 for the three branches are $\lambda_1 = \mu; -2\mu$, and -2μ, respectively. It is easy to see that $(0, 0)$ is the bifurcation point for all three branches.

The subcritical (or supercritical) pitchfork bifurcation can be described as a branch of equilibria which changes stability type at the bifurcation point is intersected there by a stable (or unstable) branch.

The following theorem establishes that the three type of bifurcations observed in the above examples are indeed the only generic ones there are.

Theorem

Let us suppose

- *(i)*$f(x,\mu)$ *is an analytic function of* (x,μ) *near* $(0,0)$.
- *(ii)* $(x,\mu)=(0,0)$ *is a bifurcation point (namely* $0=f(0,0)=\frac{\partial f}{\partial x}(0,0)$*).*

Then

- *(a) If* $\frac{\partial f}{\partial \mu}(0,0)\neq 0$ *and* $\frac{\partial^2 f}{\partial x^2}(0,0)\neq 0$, *then there exists, in a neighbourhood of* $(0,0)$, *a single branch of critical points which has a saddle node bifurcation at* $(0,0)$.
- *(b) If* $\frac{\partial f}{\partial \mu}(0,0)=0$ *let* $D=det\begin{pmatrix}\frac{\partial^2 f}{\partial \mu^2} & \frac{\partial^2 f}{\partial \mu\partial x}\\ \frac{\partial^2 f}{\partial x\partial \mu} & \frac{\partial^2 f}{\partial x^2}\end{pmatrix}(0,0)=\frac{\partial^2 f}{\partial \mu^2}\frac{\partial^2 f}{\partial x^2}-(\frac{\partial^2 f}{\partial x\partial \mu})^2$,

then (b1) if $D>0$, *then* $(0,0)$ *is an isolated critical point.*
(b2) if $D<0$, *then there are two branches of critical points which intersect at* $(0,0)$. *The bifurcation is either transcritical or pitchfork.*

Remark

It is quite possible to do more on the bifurcation theory, let's mention :

- *the one-dimensional bifurcation in two dimensions space ($n = 2$).*
- *In high dimension space there is the Hopf Bifurcation ($n \geq 2$).*
- *We invite the reader to have at hands books in which this theory is developped for instance the book of Perko.*

Thank you for your attention