Introduction to standard and non-standard Numerical Methods

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- The differential equation then becomes

$$\frac{x(t+h) - x(t)}{h} \approx f(x)$$

- If you know $x(t)$ and h then you can solve this equation for $x(t+h)$.

Solving $\frac{dx}{dt} = f(x)$

- $$\frac{x(t+h) - x(t)}{h} \approx f(x)$$

  has solution

  $$x(t+h) \approx x(t) + h \cdot f(x)$$

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- Exemple ($t = 0$): If we know $x(0)$, then this equation allows us to compute $x(0+h) = x(h)$.
- Exemple ($t = h$): If we know $x(h)$, then this equation allows us to compute $x(h+h) = x(2h)$.
- and than $x(2h+h) = x(3h)$, $x(3h+h) = x(4h)$...

Solving $\frac{dx}{dt} = f(x)$

- Pick small number of h and compute

$$
\begin{array}{ccc}
x(h) & = & x(0) + h \cdot f(0) \\
 & \searrow & | \\
x(2h) & = & x(h) + h \cdot f(h) \\
 & \searrow & | \\
x(3h) & = & x(2h) + h \cdot f(2h) \\
 & \searrow & | \\
x(4h) & = & x(3h) + h \cdot f(3h) \\
 & \vdots &
\end{array}
$$

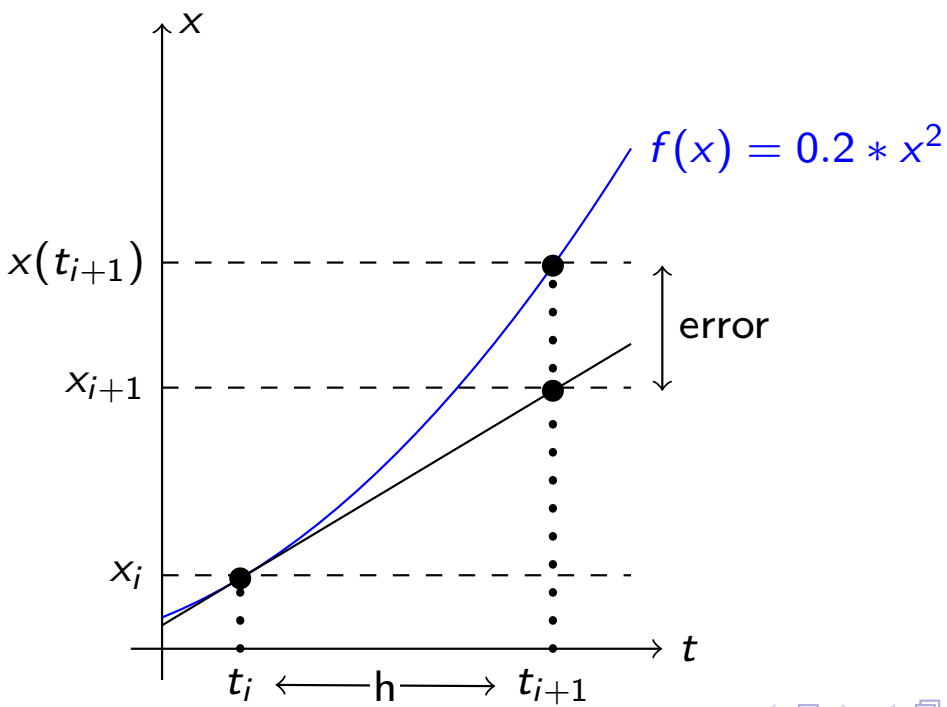
How did Euler do this?
R: By hand (main...), if yes ! How many times ?
How do we do this in the 21st century?
With computer

- Idea is that

$$\text{New value} = \text{Old value} + \text{step size*slope}$$

$$x_{n+1} = x_n + h_n f(t_n, x_n)$$

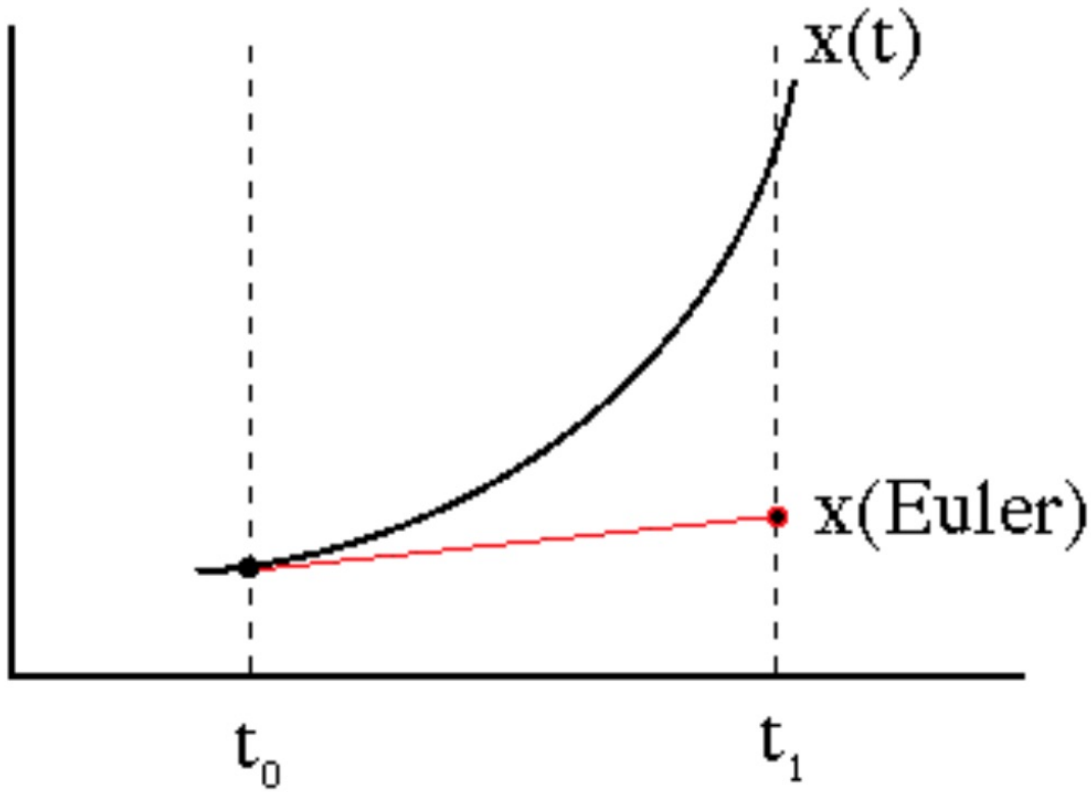
- Slope is generally a function of t, hence $x(t)$
- Different methods differ in how to estimate ϕ



Euler's Method (RK method with order 1)

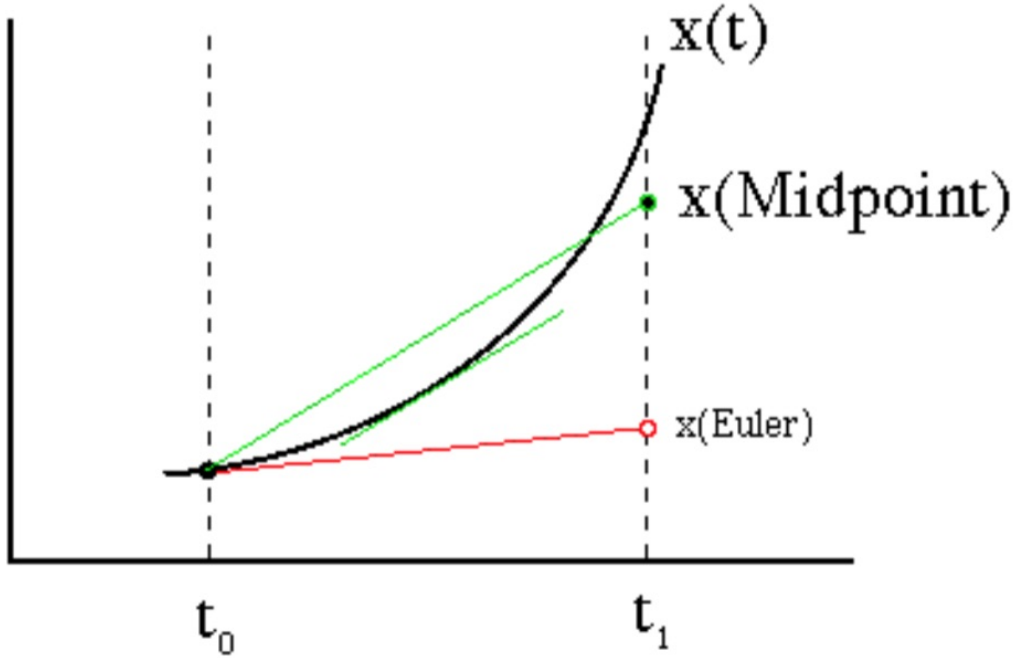
A one-step method expresses x_{n+1} in terms of the previous value x_n.
The simplest example of a one-step method for the numerical solution of the initial value problem (IVP) is Euler's method.
Given that $x(t_0) = x_0$
We define

$$x_{n+1} = x_n + h_n f(t_n, x_n)$$



The midpoint rule (RK method with order two)

$$x_{n+1} = x_n + hf(t_n + \frac{h}{2}, x_n + \frac{h}{2} f(x_n, t_n))$$



Second-Order Runge-Kutta Methods

Let

$$x^{'}(t) = f(t, x(t)).$$

$$x(t+h) = x(t) + hx^{'}(t) + \frac{h^2}{2!}x^{''}(t) + \mathcal{O}(h^3)$$

where

$$x^{''}(t) = f_t(t,x) + f_x(t,x)x^{'}(t) = f_t(t,x) + f_x(t,x)f(t,x)$$

with Jacobian f_x

$$x(t+h) = x(t) + hf(t,x) + \frac{h^2}{2!}\Big[f_t(t,x) + f_x(t,x)f(t,x)\Big] + \mathcal{O}(h^3)$$

Second-Order Runge-Kutta Methods

$$x(t+h) = x(t) + \frac{h}{2} f(t,x) + \frac{h}{2}\Big[f_t(t,x) + h f_t(t,x) + h f_x(t,x) f(t,x)\Big] + \mathcal{O}(h^3)$$

Recalling the multivariate Taylor expansion

$$f(t+h, x+k) = f(t,x) + h f_t(t,x) + f_x(t,x)k + \cdots$$

Then

$$x(t+h) = x(t) + \frac{h}{2} f(t,x) + \frac{h}{2}\Big[f(t+h, x+hf(t,x))\Big] + \mathcal{O}(h^3)$$

we get the numerical method :

$$x_{n+1} = x_n + h\Big(\frac{1}{2}k_1 + \frac{1}{2}k_2\Big)$$

with

$$k_1 = f(t_n, x_n) \quad \text{and} \quad k_2 = f(t_n + h, x_n + hk_1)$$

Fourth-Order Runge-Kutta Methods

The classical method is given by

$$x_{n+1} = x_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

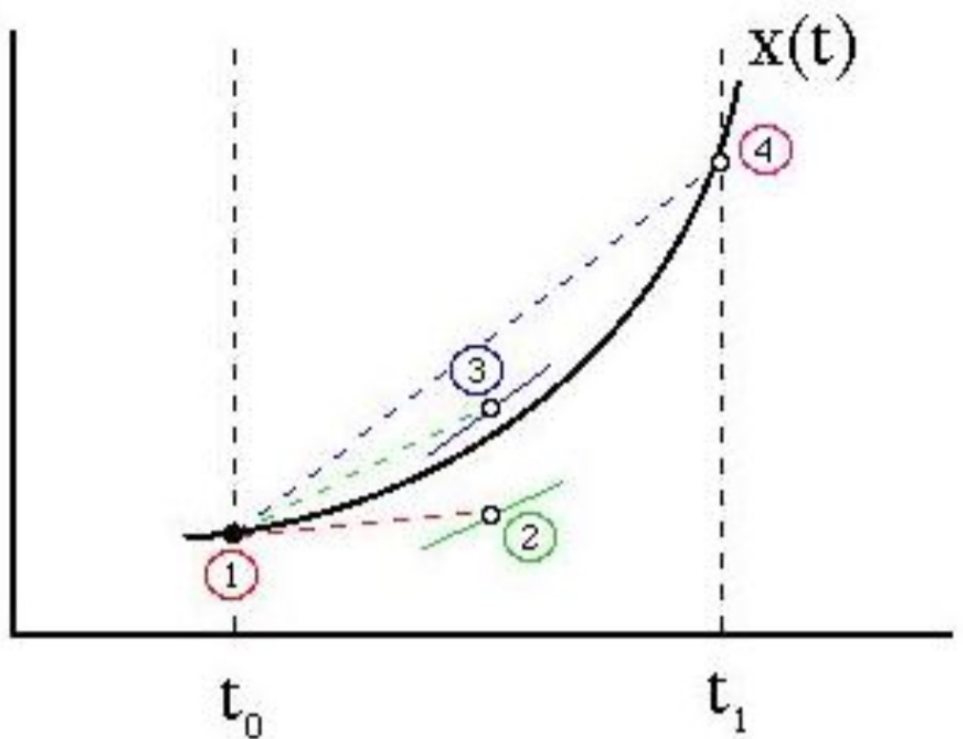
with

k_1 is the increment based on the slope at the beginning of the interval, using x (Euler's method);

k_2 is the increment based on the slope at the midpoint of the interval, using x and k_1

k_3 is again the increment based on the **slope** at the midpoint, but now using x and k_2

k_4 is the increment based on the slope at the end of the interval, using x and k_3



Butcher tableau

The family of explicit Runge-Kutta methods is a generalization of the RK4 method mentioned above. It is given by

$$x_{n+1} = x_n + h \sum_{i=1}^{s} b_i k_i$$

where

$$\begin{aligned}
k_1 &= f(t_n + 0h, x_n), \\
k_2 &= f(t_n + c_2 h, x_n + h(a_{21} k_1)), \\
k_3 &= f(t_n + c_3 h, x_n + h(a_{31} k_1 + a_{32} k_2)), \\
k_4 &= f(t_n + c_4 h, x_n + h(a_{41} k_1 + a_{42} k_2 + a_{43} k_3)), \\
&\vdots \\
k_s &= f(t_n + c_s h, x_n + h(a_{s1} k_1 + a_{s1} k_2 + \cdots + a_{s,s-1} k_{s-1})),
\end{aligned}$$

And the final step to combine these intermediate steps like this:

$$x_{n+1} = x_n + b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4 + \cdots + b_s k_s$$

The Runge-Kutta method is consistent if

$$\sum_{j=1}^{i-1} a_{ij} = c_i, \quad \text{for} \quad i = 2, \cdots, s.$$

Exemple : The explitcit RK4 method falls in this framework

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

It is not difficult to construct s-stage implicit methods which are A-stable.
For example, this can be done by choosing the coefficients c_i and b_i to be the quadrature points and weights respectively in the Gauss quadrature formula for the evaluation of

$$\int_0^1 f(t)dt \approx \sum_{i=1}^{s} b_i f(c_i)$$

The numbers a_{ij} can then be chosen so that the method has order 2s, and is A-stable.

Let us observe that on expanding $x(t_{n+1}) = x(t_n + h)$ into a Taylor series

$$x^{'}(t_n) = f(t_n, x(t_n))$$

we have that :

$$x(t_{n+1}) = x(t_n) + hf(t_n, x(t_n)) + \mathcal{O}(h^2)$$

More generally, a one-step method may be written in the form

$$x(t_{n+1}) = x(t_n) + h\phi(t_n, x_n, h), \quad n = 0, 1, \cdots, N-1, \quad x(t_0) = x_0,$$

where we assume that $\phi : [t_0, t_0 + T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.
In pratical case, the function $\phi(t, x, h)$ can be define
For example : Euler's Method is given by $\phi(t_n, x_n, h) = f(t_n, x_n)$

Global error

In order to assess the accuracy of the numerical method, we define the **global error**, e_n, by

$$e_n = x(t_n) - x_n$$

The truncation error, T_n, is define by

$$T_n = \frac{x(t_{n+1}) - x(t_n)}{h} - \phi(t_n, x(t_n); h).$$

Convergence

Consider the general one-step method where, in addition to being a continuous function of its arguments, ϕ is assumed to satisfy a Lipschitz condition with respect to its second argument, that is, there exists a positive constant L_ϕ such that, for $0 \leq h \leq h_0$ and for (t, u) and (t, v) in the rectangle

$$D = \{(t, x) : t_0 \leq x \leq t_M, \quad |x - x_0| \leq C\}$$

we have that

$$|\phi(t, u; h) - \phi(t, v; h)| \leq L_\phi |u - v|.$$

Then, assuming that $|x - x_0| \leq C$, $n = 1, 2, \cdots, N$ it follows that

$$|e_n| \leq \frac{T}{L_\phi}\left(e^{L_\phi(t_n - t_0)} - 1\right)$$

where $T = \max_{0 \leq n \leq N-1} |T_n|$

Consistance

Let us apply this general result in order to obtain a bound on the global error in Euler's method. The truncation error for Euler's method is given by

$$T_n = \frac{x(t_{n+1}) - x(t_n)}{h} - f(t_n, x(t_n)) = \frac{x(t_{n+1}) - x(t_n)}{h} - x^{'}(t_n)$$

Taylor series :

$$x(t_{n+1}) = x(t_n) + hx^{'}(t_n) + \frac{h^2}{2!}x^{''}(\zeta_n), \quad t_n < \zeta_n < t_{n+1}.$$

$$T_n = \frac{1}{2}hx^{''}(\zeta_n)$$

$$M = \max_{\zeta \in [t_0, t_M]} |x^{''}|$$

$$|T_n| \leq T = \frac{1}{2}M\left[\frac{e^{L(t_n - t_0)} - 1}{L}\right], n = 0, 1, \cdots, N.$$

Remark : In practice, for such h we shall have $|x(t_n) - x_n| = |e_n| \leq Tol$ for any $n = 0, 1, \cdots, N$. So $h \leq$ (expression $*$ Tol)

Bref definition

- A Scheme is said explicit if $x(t_{i+1})$ can be write as a linear combination of $x(t_i), f(x_k, t_{i_1}), \ldots \forall k$. The scheme is implicite if it others values are necessar.

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$$\lim_{h,\Delta t \to 0} (M\phi - M_{h,\Delta t}\phi) = 0$$

- A linear scheme consistant is convergent if only if it is stable.

Nonstandard Finite Difference Scheme (NFDS)

Exact schemes : Examples

Let

$$\frac{dx}{dt} = f(t,x) = -\lambda x \quad \text{and} \quad x(t_0) = x_0$$

The general solution is given by :

$$x(t) = x_0 e^{-\lambda(t-t_0)}.$$

then the exact scheme is

$$u_{k+1} = e^{-\lambda h} u_k.$$

NFDS : Examples

Let

$$\frac{dx}{dt} = f(t,x) = -\lambda x \quad \text{and} \quad x(t_0) = x_0$$

$$x_{k+1} = e^{-\lambda h} x_k.$$

Combining the previous exact schemes, the NFDS scheme is given by :

$$x_{k+1} - x_k = (e^{-\lambda h} - 1)x_k = -\lambda \left(\frac{1 - e^{-\lambda h}}{\lambda} \right) x_k, \tag{1}$$

than

$$\frac{x_{k+1} - x_k}{\phi(h)} = -\lambda x_k \qquad \text{avec} \qquad \phi(h) = \frac{1 - e^{-\lambda h}}{\lambda} \tag{2}$$

The Lotka-Volterra system: Euler's Method

$$\begin{cases} \frac{x_{k+1}-x_k}{\phi_1(h)} &= ax_k - bx_{k+1}y_k, \\ \frac{y_{k+1}-y_k}{\phi_2(h)} &= -cy_{k+1} + dx_{k+1}y_k, \end{cases}$$

with

$$\phi_1(h) = -\frac{1-e^{-\lambda h}}{\lambda} \quad \text{and} \quad \phi_2(h) = -\frac{1-e^{-\lambda h}}{\lambda}$$

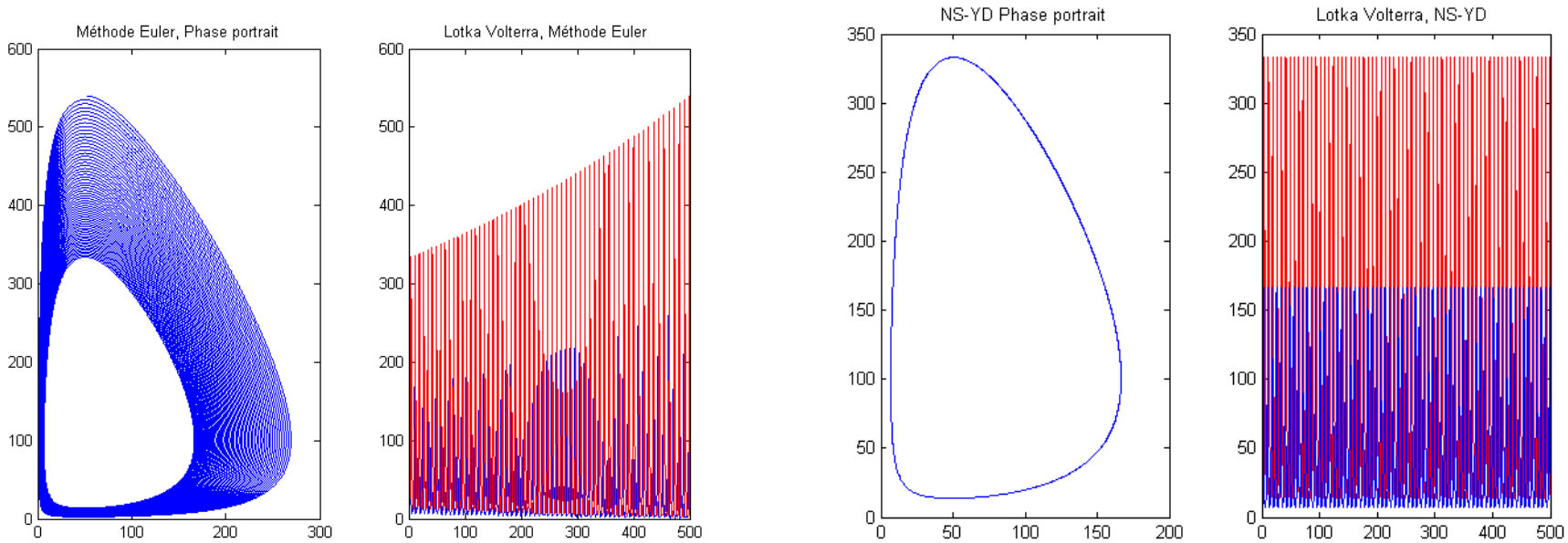

Figure: Phase plane and time history : Euler explicit method and NFDS method of Lotka-Volterra system (a=1;b=0.01;c=1;d=0.02)

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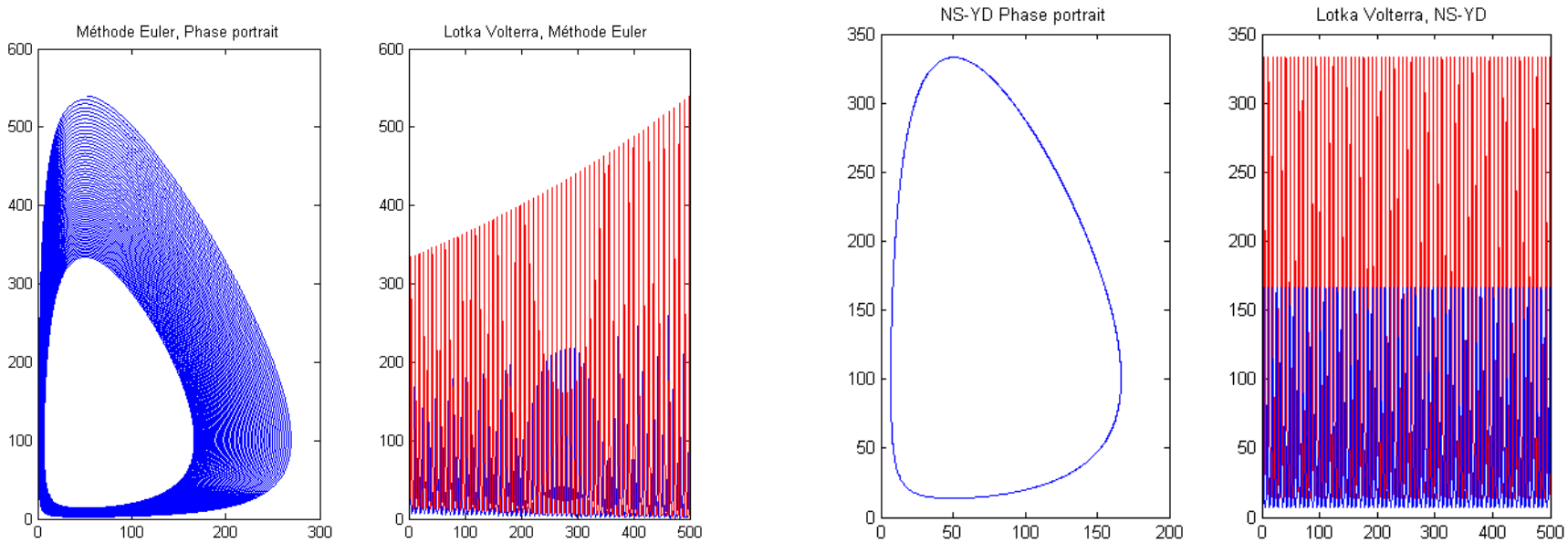

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- Rule 5 The scheme should not introduce extraneous or spurious solutions.
- Rule 6 For N differential system, it could be useful to construct nonstandard schemes for subsystems of $M < N$ differential equations and to combine them to obtain a consistant scheme.

Some examples of NFDS of ODE

Équations	NFDS	*Denominators*
$\frac{dx}{dt} = -\lambda x$	$\frac{x_{k+1}-x_k}{\phi(h)} = -\lambda x_k$	$\phi(h) = \frac{1-e^{-\lambda h}}{\lambda}$
$\frac{dx}{dt} = -\lambda x$	$\frac{x_{k+1}-x_k}{\phi(h)} = -\lambda x_{k+1}$	$\phi(h) = \frac{e^{\lambda h}-1}{\lambda}$
$\frac{dx}{dt} = x^2$	$\frac{x_{k+1}-x_k}{\phi(h)} = x_k x_{k+1}$	$\phi(h) = h$
$\frac{dx}{dt} = \lambda_1 x - \lambda_2 x^2$	$\frac{x_{k+1}-x_k}{\phi(h)} = \lambda_1 x_k - \lambda_2 x_k x_{k+1}$	$\phi(h) = \frac{e^{\lambda_1 h}-1}{\lambda_1}$
$\frac{dx}{dt} = \lambda_1 x - \lambda_2 x^2$	$\frac{x_{k+1}-x_k}{\phi(h)} = \lambda_1 x_{k+1} - \lambda_2 x_k x_{k+1}$	$\phi(h) = \frac{1-e^{-\lambda_1 h}}{\lambda_1}$
$\frac{d^2x}{dt^2} + \omega^2 x = 0$	$\frac{x_{k+1}-2x_k+x_{k-1}}{\phi^2(h)} + \omega^2 x_k = 0$	$\phi(h) = \frac{2}{\omega} sin\left(\frac{h\omega}{2}\right)$
$\frac{d^2x}{dt^2} = \lambda \frac{dx}{dt}$	$\frac{x_{k+1}-2x_k+x_{k-1}}{\phi_1(h)} = \lambda \frac{x_{k+1}-x_k}{\phi_2(h)}$	$\phi_1(h) = (\frac{e^{\lambda h}-1}{\lambda})h; \phi_2(h) = h$

Fundamental Rules?

- it is fundamental to follow rules 2 and 3. The others rules then follow...
- to build the best numerical scheme we need a deep theoretical study of the continuous problem is helpful to capture the properties of the solution and the problem

Re-formulation for applying GAS in dynamical systems: Kamgang-Sallet

$$\frac{dx}{dt} = A(x)x + f, \qquad x(0) = x_0,$$

$$\begin{cases} \frac{dS}{dt} &= (1-\pi)\rho - (\tau + \vartheta + \mu)S, \\ \frac{dV}{dt} &= \pi\rho + \tau S - (\mu + \gamma)V, \\ \frac{dI}{dt} &= \vartheta S - (\alpha + \delta + \mu + \kappa)I, \\ \frac{dJ}{dt} &= \delta I - \mu J, \\ \frac{dG}{dt} &= \xi I - \eta G, \\ \frac{dR}{dt} &= \gamma V + \alpha I - \mu R, \end{cases}$$

where

$$\vartheta = \beta_h \frac{I}{N} + \beta_G \frac{G}{G+K}, \quad y = (S, V, R), \quad z = (I, J, G)$$

$$\begin{cases} \frac{dy}{dt} &= A_1(x)(y - y^*) + A_{12}(x)z, \\ \frac{dz}{dt} &= A_2(x)z, \end{cases}$$

Application of GAS with numerical dynamical systems : Kamgang and Sallet

$$\phi(h) \leq min\left\{\frac{1}{\mu+\gamma}, \frac{1}{\alpha+\delta+\mu+\kappa}, \frac{1}{\mu}, \frac{1}{\eta}\right\}.$$

$$\exists Q \text{ such that } Q \geq \max \frac{1}{2}\{-\lambda_i\},\ i = 1, \cdots, 5,$$

chose of $\phi(h)$ is given by

$$\phi(h) = \frac{\varphi(Qh)}{Q} \text{ avec } \varphi(z) = 1 - e^{-z},\ \forall\ z \in \mathbb{R}_+.$$

Implicit scheme

$$\begin{cases} \frac{S^{k+1}-S^k}{\phi(h)} &= (1-\pi)\rho - (\tau + \beta_h \frac{I^k}{N^k} + \beta_G \frac{G^k}{G^k+K} + \mu)S^{k+1}, \\ \frac{V^{k+1}-V^k}{\phi(h)} &= \pi\rho + \tau S^{k+1} - (\mu+\gamma)V^{k+1}, \\ \frac{I^{k+1}-I^k}{\phi(h)} &= (\beta_h \frac{I^k}{N^k} + \beta_G \frac{G^k}{G^k+K})S^{k+1} - (\alpha+\delta+\mu+\kappa)I^{k+1}, \\ \frac{J^{k+1}-J^k}{\phi(h)} &= \delta I^{k+1} - \mu J^{k+1}, \\ \frac{G^{k+1}-G^k}{\phi(h)} &= \xi I^{k+1} - \eta G^{k+1}, \\ \frac{R^{k+1}-R^k}{\phi(h)} &= \gamma V^{k+1} + \alpha I^{k+1} - \mu R^{k+1}, \end{cases}$$

$$\begin{cases} \frac{y^{k+1}-y^k}{\phi(h)} &= A_1(x^k)(y^{k+1}-y^*) + A_{12}(x^k)z^{k+1}, \\ \frac{z^{k+1}-z^k}{\phi(h)} &= A_2(x^k)z^{k+1}, \end{cases}$$

explicit scheme

$$\begin{cases} \frac{S^{k+1}-S^k}{\phi(h)} &= (1-\pi)\rho - (\tau + \beta_h \frac{I^k}{N^k} + \beta_G \frac{G^k}{G^k+K} + \mu)S^{k+1}, \\ \frac{V^{k+1}-V^k}{\phi(h)} &= \pi\rho + \tau S^k - (\mu+\gamma)V^k, \\ \frac{I^{k+1}-I^k}{\phi(h)} &= (\beta_h \frac{I^k}{N^k} + \beta_G \frac{G^k}{G^k+K})S^{k+1} - (\alpha+\delta+\mu+\kappa)I^k, \\ \frac{J^{k+1}-J^k}{\phi(h)} &= \delta I^k - \mu J^k, \\ \frac{G^{k+1}-G^k}{\phi(h)} &= \xi I^k - \eta G^k, \\ \frac{R^{k+1}-R^k}{\phi(h)} &= \gamma V^k + \alpha I^k - \mu R^k, \end{cases}$$

$$\begin{cases} \frac{y^{k+1}-y^k}{\phi(h)} &= A_1(x^k)(y^k - y^*) + A_{12}(x^k)z^k, \\ \frac{z^{k+1}-z^k}{\phi(h)} &= A_2(x^k)z^k, \end{cases}$$

Simulations

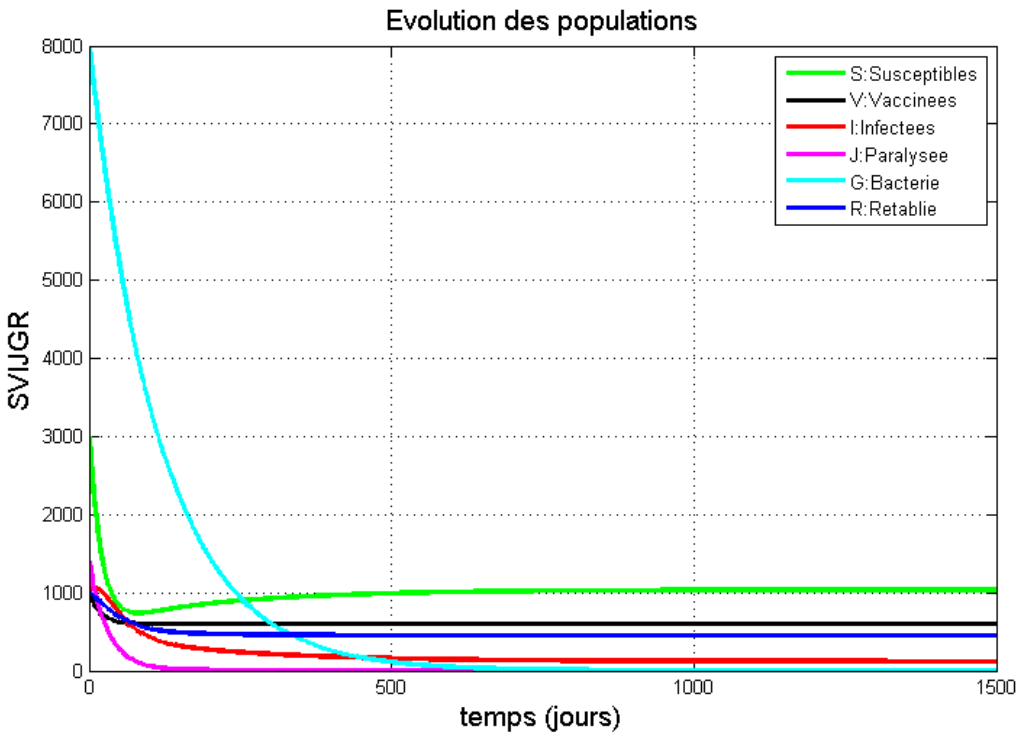

Figure: $\underline{\mathcal{R}_0 > 1.}$

Simulations

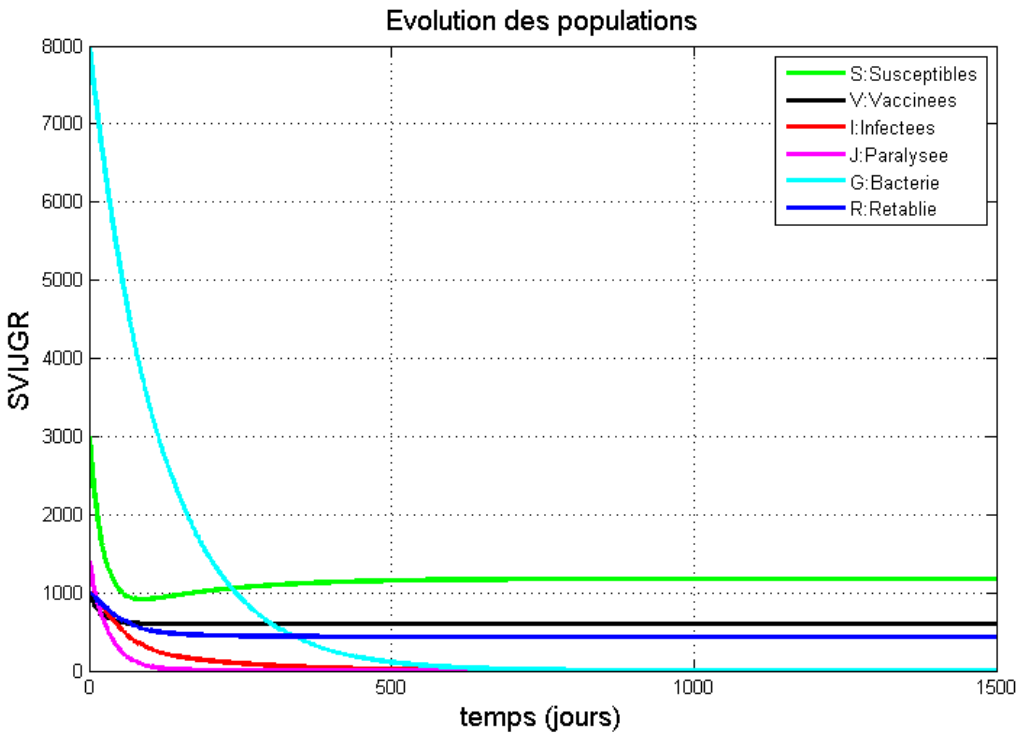

Figure: $\underline{\mathcal{R}_0 < 1.}$

Comparison

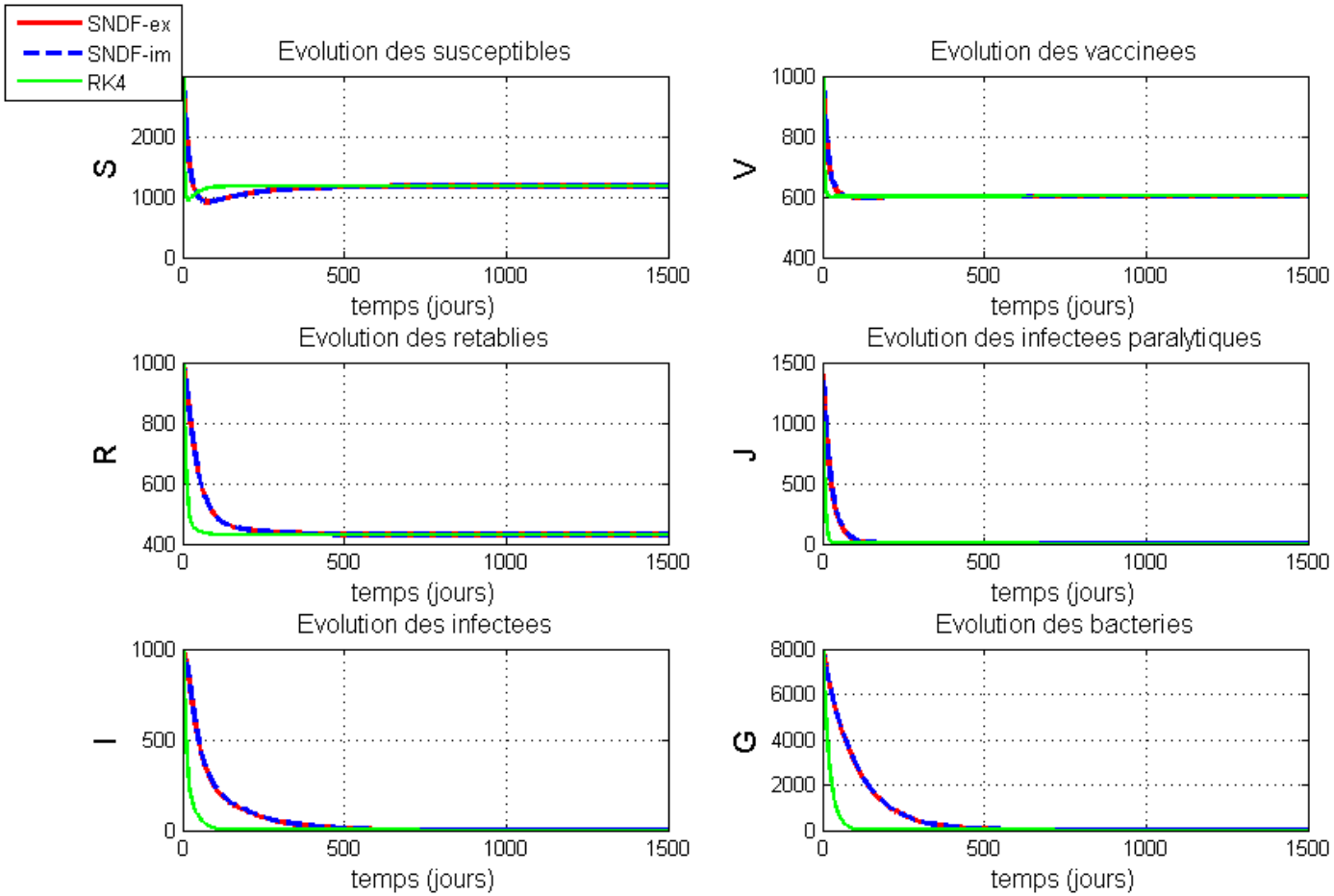

Figure: Comparison between NFDS and RK4

Comparison

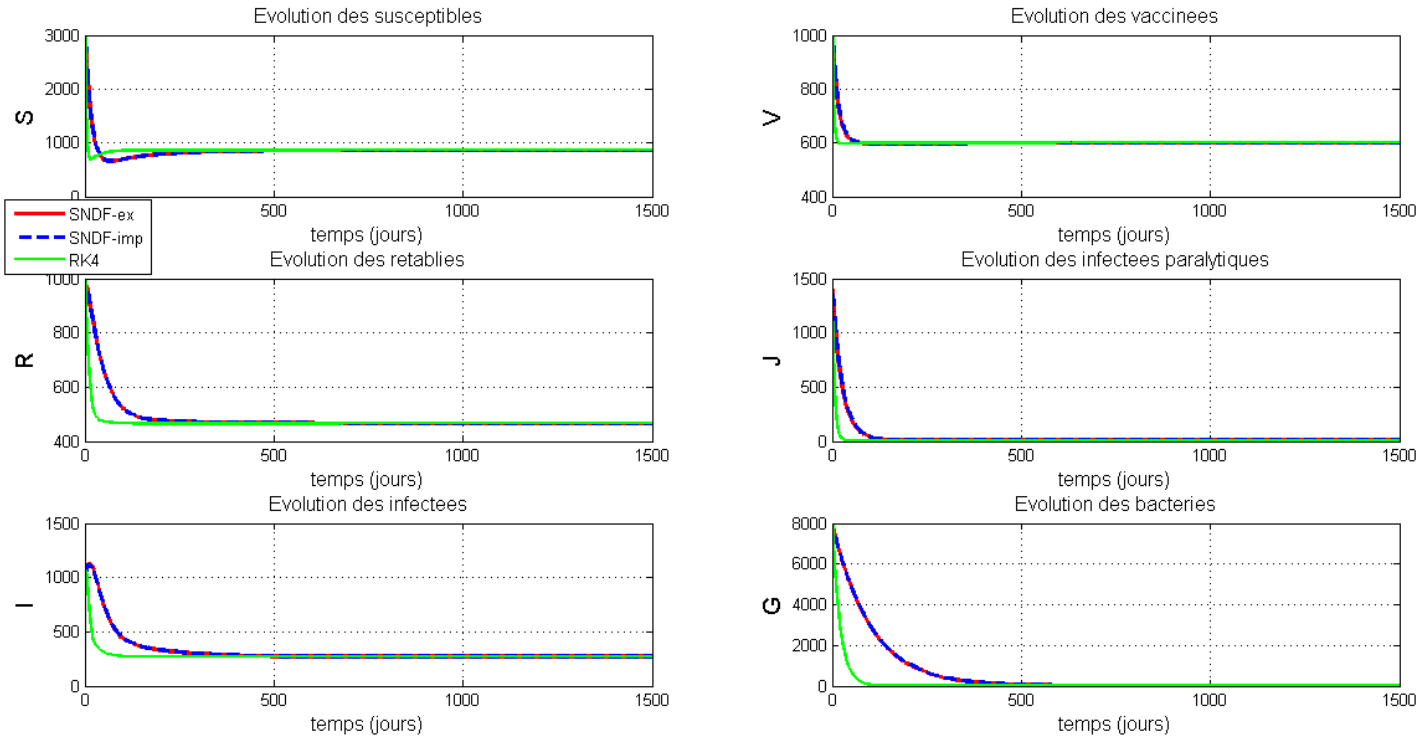

Figure: Comparison between NFDS and RK4